MA10209 Algebra 1A

Sheet 2 Problems v1: GCS

7-x-11

The course website is http://people.bath.ac.uk/masgcs/diary.html

Hand in work to your tutor by 13:00, Monday Oct 17.

1. In each case, determine whether the statement really defines a map, or it is defective in some way.
   (a) $f : \mathbb{C} \longrightarrow \mathbb{C}$ defined by $f(z) = 1/z$ for each $z \in \mathbb{C}$.
   (b) $g : \mathbb{C} \longrightarrow \mathbb{C}$ defined, for each $z \in \mathbb{C}$, by letting $g(z)$ be the complex number such that $g(z)^2 = z$.
   (c) h is the function $\cos x$.
   (d) $j : \mathbb{R} \longrightarrow \mathbb{R}$ defined, for each $x \in \mathbb{R}$, by $j(x) = \sin(\cos(\tan(x)))$.
   (e) $k : \mathbb{Q} \longrightarrow \mathbb{Q}$ by, for each $x \in \mathbb{Q}$, $k(x) = \sqrt{|x|}$ (with the convention that $\sqrt{}$ means take the non-negative square root).

2. In each case, determine which of the properties *injectivity*, *surjectivity* and *bijectivity* are enjoyed by the given function. Please give reasons.
   (a) $f_1 : \mathbb{N} \longrightarrow \mathbb{Z}$ defined by $f_1(x) = x^2$ for each $x \in \mathbb{N}$.
   (b) $f_2 : \mathbb{Z} \longrightarrow \mathbb{Z}$ defined by $f_2(x) = x^2$ for each $x \in \mathbb{Z}$.
   (c) $f_3 : \mathbb{C} \longrightarrow \mathbb{R}$ defined by $f_3(x) = |x|$ for each $x \in \mathbb{C}$.
   (d) $f_4 : \mathbb{C} \longrightarrow \{r^2 \mid r \in \mathbb{R}\}$ defined by $f_4(x) = |x|$ for each $x \in \mathbb{C}$.
   (e) $f_5 : \mathbb{N} \longrightarrow \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ defined by letting $f_5(x)$ be the leftmost digit in the ordinary base 10 (decimal) representation of n.
   (f) $f_6 : \{r \mid r \in \mathbb{R}, -\pi/2 < r < \pi/2\} \longrightarrow \mathbb{R}$ defined by letting $f_6(x) = \tan x$.

3. Let $I_n = \{1, 2, \dots n\}$ be the set which consists of the first n natural numbers, and let $S = \{0, 1\}$. In each case you should justify your answer, for a numerical response will not suffice.
   (a) How many maps f are there such that $f : I_n \longrightarrow S$?
   (b) How many surjective maps f are there such that $f : I_n \longrightarrow S$?
   (c) How many injective maps f are there such that $f : I_n \longrightarrow S$?
   (d) How many bijective maps f are there such that $f : I_n \longrightarrow I_n$?
   (e) How many surjective maps f are there such that $f : I_n \longrightarrow I_n$?
   (f) How many injective maps f are there such that $f : I_n \longrightarrow I_n$?

4. Let $f : A \longrightarrow B$ be a map. Let $X = \{f(a) \mid a \in A\} \subseteq B$. Show that there is a (natural) surjective map $g : A \longrightarrow X$ and a (natural) injective map $h : X \longrightarrow B$ such that $f = h \circ g$. *Hint: There are obvious recipes which define the maps g and h. This is what the word 'natural' means in this context.*

5. Suppose that $f : A \longrightarrow B$ and $g : B \longrightarrow C$ are maps. *In each case, you should either give a proof that the result follows, or a specific example to show that it does not follow.*

   (a) Suppose that $g \circ f$ is injective. Does it follow that f is injective?

   (b) Suppose that $g \circ f$ is injective. Does it follow that g is injective?

   (c) Suppose that $g \circ f$ is surjective. Does it follow that f is surjective?

   (d) Suppose that $g \circ f$ is surjective. Does it follow that g is surjective?

6. Suppose that $f : A \longrightarrow A$ and $g : A \longrightarrow A$. *In each case, you should either give a proof that the result follows, or a specific example to show that it does not follow.* We omit brackets from compositions of three (or more) functions since the associative law has been established.

   (a) Suppose that $g \circ f$ and $f \circ g$ are both bijective. Does it follow that f and g are both bijective?

   (b) Suppose that $f \circ f$ is bijective. Does it follow that f is bijective?

   (c) Suppose that $f \circ g \circ f$ is bijective. Does it follow that g is bijective?

7. Consider the maps $f, g : \mathbb{Z} \longrightarrow \mathbb{Z}$ defined by $f(x) = x + 1$ for each $x \in \mathbb{Z}$ and $g(x) = 2x$ for each $x \in \mathbb{Z}$.

   (a) Determine all maps $h : \mathbb{Z} \longrightarrow \mathbb{Z}$ such that $f \circ h = h \circ f$.

   (b) Determine all maps $j : \mathbb{Z} \longrightarrow \mathbb{Z}$ such that $g \circ j = j \circ g$.

   (c) Determine all maps $k : \mathbb{Z} \longrightarrow \mathbb{Z}$ such that both $f \circ k = k \circ f$ and $g \circ k = k \circ g$.

8. Exhibit (i.e. give examples of) bijections between the given sets:

   (a) Domain $\mathbb{N}$, codomain $\mathbb{Z}$.

   (b) Domain $\mathbb{N}^2$, codomain $\mathbb{Z}$.

   (c) Domain $\{r \mid r \in \mathbb{R}, 0 < r < 1\}$, codomain $\mathbb{R}$.

   (d) (interesting) Domain $\{A \mid A \subseteq \mathbb{N}, |A| < \infty\}$, codomain $\mathbb{N}$.

9. (A little trickier) Suppose that S is a finite set of size n and that $f : S \longrightarrow S$ is a bijection. Define $f^0 = \mathrm{Id}_S$ and if $m > 0$ is a positive integer, then we define f^m to be $f \circ f^{m-1}$. Prove that $f^{n!} = \mathrm{Id}_S$, the identity map from S to S.

10. *Tutor pacifier. For enthusiasts only.* Prove that there exist two functions $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ such that $f \circ g$ is strictly decreasing and $g \circ f$ is strictly increasing.