MA10209 Algebra and Geometry

Sheet 1 Solutions v0: GCS

1-x-11

1. Find an economical way to write these sets:

   (a) $\{u \mid u \in \mathbb{Z}, u^2 - 2u + 1\}$.
   Solution This set is $\{1\}$. It can be written less economically as $\{1, 1\}$ because $\{1\} = \{1, 1\}$. This is a tempting (but pointless) description because 1 is a "double root" of $(u - 1)^2$.

   (b) $\{v \mid v \in \mathbb{Z}, v^3 - 6v^2 + 11v - 6 = 0\}$.
   Solution This set is $\{1, 2, 3\}$ because the polynomial expression on the left of the equation is $(v - 1)(v - 2)(v - 3)$, and this takes the value 0 if, and only if, v is 1, 2 or 3.

   (c) $\{w \mid w \in \mathbb{R}, r^2 + 1 = 0\}$.
   Solution This set is $\emptyset$ (the empty set) because squares of real numbers are never negative.

   (d) $\{x \mid x \in \mathbb{Q}, x^2 \in \mathbb{Z}\}$. In this case you may not be able to prove your answer, but at least make an educated guess as to what this set actually is.
   Solution This set X is economically written as $\mathbb{Z}$. Most students will not actually have the machinery to prove this result yet. All the necessary techniques will be developed in this course.

   Clearly $\mathbb{Z} \subseteq X$. Now suppose that $x \in X$, so $x = a/b$ for some integers a and b where $b \neq 0$. By cancelling common factors, we may assume that the greatest common divisor of a and b is 1. If $b = \pm 1$, the proof is complete. Now we suppose, for contradiction (and hoping to be proved wrong), that $b \notin \{-1, 1\}$. Choose a prime p which divides b (but is there such a thing?). Now $xb^2 = a^2$ so p divides a^2 so (and this may be 'obvious' but it will need proof) p divides a. Now p is a common divisor of a and b and $p > 1$. This is the required contradiction, because g.c.d.$(a, b) = 1$.

   (e) $\{y \mid y \in \emptyset \cap \mathbb{N}\}$.
   Solution There are no elements in the empty set, so this set is $\emptyset$.

   (f) $\{z \mid z \in \mathbb{C}, z^2 + 1 = 0\}$.
   Solution Notice that $z^2 + 1 = (z - i)(z + i)$ and this vanishes if, and only if, $z = i$ or $z = -i$. Therefore this set is $\{i, -i\}$.

2. Suppose that A, B and C are finite sets. Suppose that $|A|, |B|, |C|, |A \cap B|, |B \cap C|, |C \cap A|$ and $|A \cap B \cap C|$ are given. Does this information determine $|A \cup B \cup C|$? If so, how and why?
   Solution Look at the sum

$$|A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|.$$

   each element of either A, B or C which is not in another of thesese three sets contributes 1 in this expression. An element which is in two but not in all three of A, B and C contributes $1+1-1 = 1$ too. Finally an element which is in all three sets contributes $1 + 1 + 1 - 1 - 1 - 1 + 1 = 1$ also. Therefore this expression is the value of $|A \cup B \cup C|$. There is a generalization which enables you to calculate the size of the union of k finite sets, given that you know the sizes of the sets, and the sizes of all intersections. This is the *inclusion-exclusion principle*, and tells you that the size of a union is the sum of the sizes of the sets, minus the sizes of all possible intersection in pairs, plus the sizes of all possible intersections of three, minus etc etc. There are several proofs available, including a modification of the proof given here when $k = 3$. The solution to 9(b) is helpful in this context.

3. Let $A = \{1, 2, 3\}$ and $B = \{3, 4\}$ be subsets of the integers. In each case determine if there is a set $X \subseteq \mathbb{Z}$ which satisfies the given conditions, and discuss whether or not this is the unique subset of $\mathbb{Z}$ which satisfies the conditions.

   (a) $X \cap B = \emptyset$ and $A \subseteq X \cup B$.
   Solution $X = \{1, 2\}$ will do nicely, as will any subset of $\mathbb{Z}$ which is disjoint from B but is a superset of $\{1, 2\}$. For example, $X' = \{1, 2, 1729\}$ is also a solution.

   (b) $X \cup B = A$ and $X \cup A = B$.
   Solution There is no such set X, because if there were such a set X, then $A \subseteq X \cup A = B$ which is false.

   (c) $X \cup X = A$ and $|X \cap B| = 1$.
   Solution The only candidate for a solution is A because $X \cup X = X$. Now $A \cap B = \{3\}$ does have size 1, so this is the unique solution.

   (d) $X \cap X = B$ and $B \cap B = X$.
   Solution Notice that $X \cap X = X$ so B is the unique candidate for a solution. Happily B also satisfies the second condition, so indeed B is the unique solution.

4. Suppose that $f : \mathbb{R} \longrightarrow \mathbb{R}$ is defined by $f(x) = 2x$ for each $x \in \mathbb{R}$ and $g : \mathbb{R} \longrightarrow \mathbb{R}$ is defined by $f(x) = x/2$ for each $x \in \mathbb{R}$. Recall that $f \circ g : \mathbb{R} \longrightarrow \mathbb{R}$ is defined by $(f \circ g)(x) = f(g(x))$ for each $x \in \mathbb{R}$. Prove that $f \circ g = g \circ f = \mathrm{Id}_{\mathbb{R}}$ (the identity map from $\mathbb{R}$ to $\mathbb{R}$.
   Solution Suppose that $y \in \mathbb{R}$, then $(f \circ g)(y) = f(g(y)) = f(y/2) = 2(y/2) = y$. This is true for each $y \in \mathbb{R}$ and so $f \circ g = \mathrm{Id}_{\mathbb{R}}$ is the identity map from $\mathbb{R}$ to $\mathbb{R}$. Similarly, suppose that $z \in \mathbb{R}$. Now $(g \circ f)(z) = g(f(z)) = g(2z) = (2z)/2 = z$. This is true for each $z \in \mathbb{R}$ and so $g \circ f = \mathrm{Id}_{\mathbb{R}}$ is the identity map from $\mathbb{R}$ to $\mathbb{R}$.

5. Suppose that $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ is defined by $f(x) = 2x$ for each $x \in \mathbb{Z}$.

   (a) Show that there is at least one map $g : \mathbb{Z} \longrightarrow \mathbb{Z}$ such that $g \circ f = \mathrm{Id}_{\mathbb{Z}}$.
   Solution Define g by $g(m) = m/2$ if m is even, and $g(m) = 1729$ if m is odd.

   (b) Show that there are infinitely many maps $g : \mathbb{Z} \longrightarrow \mathbb{Z}$ such that $g \circ f = \mathrm{Id}_{\mathbb{Z}}$.
   Solution Define $g(m) = m/2$ when m is even, and define g arbitrarily when m is odd (and if you like, pick different values of $g(m)$ as odd m varies). However, it is (stylistically) better to be as specific as possible. Thus an excellent solution is to say, choose any integer k, and define $g(m) = k$ whenever m is odd. This defines a different g such that $g \circ f = \mathrm{Id}_{\mathbb{Z}}$ for each $k \in \mathbb{Z}$, and so provides infinitely many such functions.

   (c) Show that there are no maps $h : \mathbb{Z} \longrightarrow \mathbb{Z}$ such that $f \circ h = \mathrm{Id}_{\mathbb{Z}}$.
   Solution The function f only takes even values, so $f \circ h$ only takes even values, irrespective of the function h. For any h, $(f \circ h)(1)$ is even, and so is not 1. Thererefore $f \circ h$ cannot be the identity map on $\mathbb{Z}$ (i.e. from $\mathbb{Z}$ to $\mathbb{Z}$).

   (d) (interesting!) Is there a map $k : \mathbb{Z} \longrightarrow \mathbb{Z}$ such that $k \circ k = f$?
   Solution This problem demands creative thinking. Tutors please be circumspect about giving too much away too soon. Here is a possible solution. $f : 0 \mapsto 0$. Now suppose that u is an odd positive integer. Require that $f : u \mapsto -u \mapsto 2u \mapsto -2u \mapsto 4u \mapsto -4u \mapsto \cdots$ and $f : 0 \mapsto 0$. If x is a positive integer, then $f(x) = -x$ and if y is a negative integer, then $f(y) - 2y$. Now, is there a map $l : \mathbb{Z} \longrightarrow \mathbb{Z}$ such that $l \circ l \circ l = f$?

6. If A is a set, then $A^2 = A \times A$ is the set with elements the ordered pairs (x, y) where $x, y \in A$. René Descartes (1596-1650) invented the rectangular co-ordinate system which sets up a bijection between a geometric plane and the set $\mathbb{R}^2$. Analogous things happen in 3-dimensions. In each case, give a geometric meaning to the set or map.

   (a) $\{(x, y) \mid (x, y) \in \mathbb{R}^2, x^2 + y^2 = 1\}$
   Solution This is a unit circle (i.e. it has radius 1) and its centre is the chosen origin.

   (b) $\{(x, y) \mid (x, y) \in \mathbb{R}^2, x^2 + y^2 = -1\}$
   Solution This is $\emptyset$, the empty set of points in the plane.

(c) $\{(x,y) \mid (x,y) \in \mathbb{R}^2, 3x+4y=5\}$
Solution This is the straight line through $(\frac{3}{5}, \frac{4}{5})$ which is perpendicular to the vector $(\frac{3}{5}, \frac{4}{5})$. It therefore has slope $-\frac{3}{4}$. (Note the dot product (scalar product) interpretation of $(\frac{3}{5}, \frac{4}{5}) \cdot (x,y) = 1$, and the fact that $(\frac{3}{5}, \frac{4}{5})$, viewed as a vector, is a *unit vector*.)

(d) $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ where $f : (x,y) \mapsto (y,x)$ for each $(x,y) \in \mathbb{R}^2$.
Solution This map is reflection in the line with equation $y - x = 0$.

(e) $g : \mathbb{R}^2 \mapsto \mathbb{R}^2$ where $g : (x,y) \longrightarrow (-x,-y)$ for each $(x,y) \in \mathbb{R}^2$.
Solution There is no easy way to say this in English. Some people say that this is reflection in the origin, but that phrasing is not universally accepted English usage. Less controversially one can say that it is an enlargement (scaling, dilation, homothety) from the origin with scale factor -1. Alternatively this map is rotation through π about the origin.

(f) $\{(x,y,z) \mid (x,y,z) \in \mathbb{R}^3, x+y+z=1, x \geq 0, y \geq 0, z \geq 0\}$.
Solution This is a solid equilateral triangle with vertices at $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$. Persons wishing to use Cartesian co-ordinates to study a regular tetrahedron would be well advised to work in $\mathbb{R}^4$, because by paying the price of an extra dimension, the co-ordinates of the vertices become far more tractable. Similar remarks apply in higher dimensions.

(g) $\{(x,y,z) \mid (x,y,z) \in \mathbb{R}^3, x^2+y^2+z^2 < 1\}$
Solution This is a *ball* without its enclosing *sphere*. It has centre the origin and radius 1. Notice that mathematicians distinguish between a ball and a sphere, just as they distinguish between a circle and a disk.

(h) $\{(x,y,z) \mid (x,y,z) \in \mathbb{R}^3, (x-1)^2+(y+1)^2+z^2 < 1\}$
Solution This is a ball (without its enclosing sphere). Its centre is at $(1,-1,0)$ and its radius is 1.

7. A finite set P of people meet at a social occasion. Some of them shake hands with one another, and no person shakes hands with the same person more than once. Let the set of handshakes S. Let $A \subset P \times S$ consist of all pairs (p,s) where the person p takes part in the handshake s.

(a) Show that $|A|$ is an even number.

(b) Deduce that the number of people who shake hands an odd number of times was even.
Solution This method is called *double counting*, and is an effective method to extract sunlight from cucumbers. You count the size of the set A in two different ways, and equate the answers. Each handshake s involves two people, so the size of the set A must be even. Let there be e people who shake hands an even number of times, and o people who shake hands an odd number of times. If a given person p shakes hands an even number of times, there are an even number of pairs (p,x) in A. If a given person q shakes hands an odd number of times, there are an odd number of pairs (q,x) in A Since $|A|$ is even, the number of people who shake hands an odd number of times must be even (because the sum of an odd number of odd numbers is odd).

8. Suppose that $f : A \longrightarrow B$ is a map. We define the *graph* of f to be $\text{Graph}(f) = \{(a, f(a)) \mid a \in A\}$ so $\text{Graph}(f) \subseteq A \times B$.

(a) Suppose that $f : \mathbb{R} \longrightarrow R$ is defined by $f(x) = x^2$ for each $x \in \mathbb{R}$. Using Descartes's identification of $\mathbb{R}^2$ with the geometric plane, draw a picture of Graph(f).
Solution This is a familiar nose down parabola which passes through the origin and is symmetric about the y-axis.

(b) Suppose that f and g are both maps from A to B and that $\text{Graph}(f) = \text{Graph}(g)$. Does it follow that $f = g$?
Solution yes it does. Suppose that $\text{Graph}(f) = \text{Graph}(g) = G$. Then for each $a \in A$, there is a unique $c \in B$ such that $(a,c) \in G$. Therefore $f(a) = c = g(a)$.

(c) Suppose that A and B are both finite sets, and that $|A| = a$ and $|B| = b$. How many subsets G of $A \times B$ are graphs of functions from A to B?

Solution We have shown in part (b) that each map (function) from A to B gives rise to a different graph, so the issue is, how many such maps are there? For each $a \in A$, we can define $f(a) \in B$ in any one of $|B| = b$ ways, and the choice we make for any given $a \in A$ does not constrain any other choices we make. Therefore there are $b \times b \times \cdots \times b = b^a$ maps, and so b^a graphs.

9. Recall that $\binom{n}{r}$ is the number of different subsets of size r that you can find in a set of size n.

   (a) Show that $\sum_{r=0}^{n} \binom{n}{r} = 2^n$.
   Solution You can either observe that we are counting how many subsets there are of a set of size n in two ways (we are double-counting again), or use the binomial expansion of $(1+1)^n$. The details of the double counting are as follows: you can make a subset of the given set of size n by looking at each of its elements in turn, and either accepting or rejecting that element as an element of a subset. The subsets therefore correspond to lists of n binary choices, and there 2^n such sequences of choices. On the other hand you can count the number of subsets of size r as $\binom{n}{r}$, and sum over all relevant r.

   (b) Show that $\sum_{r=0}^{n} (-1)^r \binom{n}{r} = 0$ if $n > 0$.
   Solution This time the binomial expansion of $(1-1)^n$ is clearly the slickest solution. Note the special status of $n = 0$, because in combinatorics (counting arguments) it is necessary to define $0^0 = 1$. In other parts of mathematics, 0^0 can be controversial.

   (c) Suppose that $n = 2m > 0$ is an even integer. Prove that $\sum_{t=0}^{m} \binom{n}{2t} = 2^{n-1}$.
   Solution Add the previous two answers, and divide by 2.

10. *This is a very hard problem, and is here to entertain the most enthusiastic students (and perhaps tutors). Feel under no obligation even to read this problem.* Twenty-one girls and twenty-one boys took part in a mathematical contest. Each contestant solved at most six problems. For each girl and each boy, at least one problem was solved by both of them. Prove that there was a problem that was solved by at least three girls and at least three boys.
    Solution Not yet. Enjoy!