MA10209 Algebra 1A

Sheet 1 v2: GCS

1-x-11

Hand in work to your tutor by 13:00, Monday Oct 10.

1. Find an economical way to write these sets:
   (a) $\{u \mid u \in \mathbb{Z}, u^2 - 2u + 1 = 0\}$.
   (b) $\{v \mid v \in \mathbb{Z}, v^3 - 6v^2 + 11v - 6 = 0\}$.
   (c) $\{w \mid w \in \mathbb{R}, w^2 + 1 = 0\}$.
   (d) $\{x \mid x \in \mathbb{Q}, x^2 \in \mathbb{Z}\}$. Even if you cannot supply a justification of your answer, at least make an educated guess as to what this set actually is.
   (e) $\{y \mid y \in \emptyset \cap \mathbb{N}\}$.
   (f) $\{z \mid z \in \mathbb{C}, z^2 + 1 = 0\}$.

2. Suppose that A, B and C are finite sets. Suppose that $|A|, |B|, |C|, |A \cap B|, |B \cap C|, |C \cap A|$ and $|A \cap B \cap C|$ are given. Does this information determine $|A \cup B \cup C|$? If so, how and why?

3. Let $A = \{1, 2, 3\}$ and $B = \{3, 4\}$ be subsets of the integers. In each case determine if there is a set $X \subseteq \mathbb{Z}$ which satisfies the given conditions, and discuss whether or not this is the unique subset of $\mathbb{Z}$ which satisfies the conditions.
   (a) $X \cap B = \emptyset$ and $A \subseteq X \cup B$.
   (b) $X \cup B = A$ and $X \cup A = B$.
   (c) $X \cup X = A$ and $|X \cap B| = 1$.
   (d) $X \cap X = B$ and $B \cap B = X$.

4. Suppose that $f : \mathbb{R} \longrightarrow \mathbb{R}$ is defined by $f(x) = 2x$ for each $x \in \mathbb{R}$ and $g : \mathbb{R} \longrightarrow \mathbb{R}$ is defined by $g(x) = x/2$ for each $x \in \mathbb{R}$. Recall that $f \circ g : \mathbb{R} \longrightarrow \mathbb{R}$ is defined by $(f \circ g)(x) = f(g(x))$ for each $x \in \mathbb{R}$. Prove that $f \circ g = g \circ f = \mathrm{Id}_{\mathbb{R}}$ (the identity map from $\mathbb{R}$ to $\mathbb{R}$).

5. Suppose that $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ is defined by $f(x) = 2x$ for each $x \in \mathbb{Z}$.
   (a) Show that there is at least one map $g : \mathbb{Z} \longrightarrow \mathbb{Z}$ such that $g \circ f = \mathrm{Id}_{\mathbb{Z}}$.
   (b) Show that there are infinitely many maps $g : \mathbb{Z} \longrightarrow \mathbb{Z}$ such that $g \circ f = \mathrm{Id}_{\mathbb{Z}}$.
   (c) Show that there are no maps $h : \mathbb{Z} \longrightarrow \mathbb{Z}$ such that $f \circ h = \mathrm{Id}_{\mathbb{Z}}$.
   (d) (interesting!) Is there a map $k : \mathbb{Z} \longrightarrow \mathbb{Z}$ such that $k \circ k = f$?

6. If A is a set, then $A^2 = A \times A$ is the set with elements the ordered pairs (x, y) where $x, y \in A$. René Descartes (1596-1650) invented the rectangular co-ordinate system which sets up a bijection between a geometric plane and the set $\mathbb{R}^2$. Analogous things happen in 3-dimensions. In each case, give a geometric meaning to the set or map.
   (a) $\{(x, y) \mid (x, y) \in \mathbb{R}^2, x^2 + y^2 = 1\}$
   (b) $\{(x, y) \mid (x, y) \in \mathbb{R}^2, x^2 + y^2 = -1\}$
   (c) $\{(x, y) \mid (x, y) \in \mathbb{R}^2, 3x + 4y = 5\}$

(d) $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ where $f : (x, y) \longrightarrow (y, x)$ for each $(x, y) \in \mathbb{R}^2$.

(e) $g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ where $g : (x, y) \longrightarrow (-x, -y)$ for each $(x, y) \in \mathbb{R}^2$.

(f) $\{(x, y, z) \mid (x, y, z) \in \mathbb{R}^3, x + y + z = 1, x \geq 0, y \geq 0, z \geq 0\}$.

(g) $\{(x, y, z) \mid (x, y, z) \in \mathbb{R}^3, x^2 + y^2 + z^2 < 1\}$

(h) $\{(x, y, z) \mid (x, y, z) \in \mathbb{R}^3, (x - 1)^2 + (y + 1)^2 + z^2 < 1\}$

7. A finite set P of people meet at a social occasion. Some of them shake hands with one another, and no person shakes hands with the same person more than once. Let the set of handshakes S. Let $A \subseteq P \times S$ consist of all pairs (p, s) where the person p takes part in the handshake s.

   (a) Show that $|A|$ is an even number.

   (b) Deduce that the number of people who shake hands an odd number of times is even.

8. Suppose that $f : A \longrightarrow B$ is a map. We define the *graph* of f to be $\text{Graph}(f) = \{(a, f(a)) \mid a \in A\}$ so $\text{Graph}(f) \subseteq A \times B$.

   (a) Suppose that $f : \mathbb{R} \longrightarrow \mathbb{R}$ is defined by $f(x) = x^2$ for each $x \in \mathbb{R}$. Using Descartes's identification of $\mathbb{R}^2$ with the geometric plane, draw a picture of $\text{Graph}(f)$.

   (b) Suppose that f and g are both maps from A to B and that $\text{Graph}(f) = \text{Graph}(g)$. Does it follow that $f = g$?

   (c) Suppose that A and B are both finite sets, and that $|A| = a$ and $|B| = b$. How many subsets G of $A \times B$ are graphs of functions from A to B?

9. Recall that $\binom{n}{r}$ is the number of different subsets of size r that you can find in a set of size n.

   (a) Show that $\sum_{r=0}^{n} \binom{n}{r} = 2^n$.

   (b) Show that $\sum_{r=0}^{n} (-1)^r \binom{n}{r} = 0$ if $n > 0$.

   (c) Suppose that $n = 2m > 0$ is an even integer. Prove that $\sum_{t=0}^{m} \binom{n}{2t} = 2^{n-1}$.

10. *This is a very hard problem, and is here to entertain the most enthusiastic students (and perhaps tutors). Feel under no obligation even to read this problem.* Twenty-one girls and twenty-one boys took part in a mathematical contest. Each contestant solved at most six problems. For each girl and each boy, at least one problem was solved by both of them. Prove that there was a problem that was solved by at least three girls and at least three boys.