# MA10209 Algebra 1A

Vacation Problems v0: GCS

14-xii-11

The course website is http://people.bath.ac.uk/masgcs/diary.html

*This work is not to be handed in.*

1. Search the internet on terms such as *Platonic solids* and *mathematical origami* to find many sites which will teach you how to make Platonic solids from paper. Paper models of Platonic solids make excellent baubles when decorating a tree.

2. Think deep thoughts about the group of orientation preserving rigid symmetries of a cube. In lectures you were told that this group was a copy of $S_4$, and that the natural set of four things being permuted is the collection of grand (or great) diagonals. Make a cube (from hard cheese or raw potato) and indicate the grand diagonals by means of cocktail sticks, labelled from 1 to 4. Is it really true that there are elements of this group corresponding to each possible permutation of these four things? Is it also true that if two orientation preserving rigid symmetries of a cube permute the cocktail sticks in the same way, then they are the same element?

3. Consider the matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}$ and, as on sheet 8, use $A$ to define a linear map $f_A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$. Now define a new map $\widehat{f} : \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2$, by restricting the domain of $f_A$. Put another way, if $z \in \mathbb{Z}^2$ we define $\widehat{f}(z)$ to be $f_A(z)$.

   (a) Prove that $\widehat{f}(z) : \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2$ is a homomorphism of groups (using co-ordinatewise addition).

   (b) Calculate $\widehat{f}((1,0))$ and $\widehat{f}((0,1))$.

   (c) Make festive decorative paper as follows. Use a white piece of paper equipped with a Cartesian co-ordinates. Colour points $P$ with co-ordinates $(x,y) \in \mathbb{Z}^2$ red if $(x,y) \in \operatorname{Im} \widehat{f}$. Colour points $Q$ with co-ordinates $(u,v) \in \mathbb{Z}^2$ blue if $(u,v) \notin \operatorname{Im} \widehat{f}$.

   (d) Observe that $\widehat{f}$ is not surjective, even though $f_A$ is bijective.

   (e) Is the determinant of $A$ somehow visible in the decorations?

4. Take a regular heptagon (use the vertices of a 50 pence coin) and label its vertices anticlockwise with the numbers 1 through to 7. There are 14 rigid symmetries of the set of vertices, each of which can be regarded as an element of $S_7$. For example, rotation about the centre through $2\pi/7$ is denoted (1,2,3,4,5,6,7). It is also $(2,3,4,5,6,7,1)$ but that is just a different name for the same map. Alternatively, you could turn the coin over about an axis through 1 and its centre. This rigid motion is denoted $(2,7)(3,6)(4,5)$. Write down all 14 elements of this group in cycle notation.

5. The group in Problem 4 is called the *dihedral group* of order 14. We will call it $D_{14}$ (but some call it $D_7$). Given any $n \geq 3$, there is a corresponding group of rigid symmetries of a regular $n$-gon (an equilateral triangle is a regular 3-gon and a square is a regular 4-gon). Using a numbering convention similar to that in Problem 4, write down the eight elements of $D_8$ in cycle notation.

6. Consider a rectangle which is not a square.

   (a) Use the an anticlockwise numbering system for its vertices to write down the elements of $V$, its group of rigid symmetries. There should be four of them (you are allowed to turn the rectangle over).

(b) Write down the multiplication table of $V$.

7. Find all the subgroups of $D_8$. You may find that Lagrange's Theorem is very helpful when doing this analysis.

8. Let $G$ be a group. For each $x \in G$, define a map $\tau_x : G \longrightarrow G$ by $\tau_x(g) = xgx^{-1}$.

   (a) Prove that each $\tau_x$ is a homomorphism.

   (b) Prove that each $\tau_x$ is a bijective map.

   (c) Conclude that each $\tau_x \in \text{Aut }(G)$, i.e. that each $\tau_x$ is an automorphism of $G$.

   (d) Consider the map $\tau : G \longrightarrow \text{Aut }(G)$ defined by $x \mapsto \tau_x$ for each $x \in G$. Prove that $\tau$ is a homomorphism.

   (e) Prove that $\text{Ker } \tau = \{y \mid y \in G, yz = zy \text{ for all } z \in G\}$.

9. Let $S_5 = \text{Sym}(5)$ denote the *symmetric* group on $\Omega_5 = \{1, 2, 3, 4, 5\}$. Thus $S_5$ consists of the all the permutations of $\Omega_5$. Give an example of an element of $S_5$ of each possible cycle shape, and work out how many elements there are of each cycle shape (making sure that your answers add to $120 = |S_5|$.

10. The group $A_n$ studied in lectures is often called the *alternating group*. Let $A_5 = \text{Alt}(5)$ denote the alternating group on $\Omega_5 = \{1, 2, 3, 4, 5\}$. Thus $A_5$ consists of the *even* permutations of $\Omega_5$. Give an example of an element of $A_5$ of each possible cycle shape, and work out how many elements there are of each cycle shape (making sure that your answers add to $60 = |A_5|$.

11. We use a bar to denote complex conjugation. Suppose that $\alpha$ and $\beta$ are complex numbers.

    (a) Prove that $\overline{\alpha + \beta} = \overline{\alpha} + \overline{\beta}$.

    (b) Prove that $\overline{\alpha - \beta} = \overline{\alpha} - \overline{\beta}$.

    (c) Prove that $\overline{\alpha \cdot \beta} = \overline{\alpha} \cdot \overline{\beta}$.

    (d) Suppose that $\beta \neq 0$. Prove that $\overline{\alpha / \beta} = \overline{\alpha} / \overline{\beta}$.

    (e) Suppose that $|\alpha| = 1$. Prove that $\overline{\alpha} = \alpha^{-1}$.

12. Let $ABCD$ be a 4-gon in the plane (i.e. a quadrilateral), labelled anticlockwise. Each of the four sides of $ABCD$ is also the side of a square, and as you walk around this 4-gon, the four squares associated with the sides are always on the right. Let the centres of the four squares be $P, Q, R$ and $S$ in anticlockwise order, so $P$ is opposite $R$ and $Q$ is opposite $S$.

    Prove that the line segments $PR$ and $QS$ have the same length and are perpendicular. *Placing the diagram in the complex plane is very helpful.*

13. In lectures we discussed the fact that that the following maps from $\mathbb{C}$ to $\mathbb{C}$ are isometries when $\mathbb{C}$ is identified with the Argand diagram: *add a fixed complex number*, *multiply by a complex number of modulus 1* and *complex conjugation*. Composing isometries is, of course, an isometry. We also discussed the fact that any triangle is similar to one in the Argand diagram with vertices at $a, b, c \in \mathbb{C}$ with $|a| = |b| = |c| = 1$, and that in fact we can arrange that $abc = 1$.

    (a) Consider the map $\theta : \mathbb{C} \longrightarrow \mathbb{C}$ defined by $\theta(z) = b + c - bc\overline{z}$ for each $z \in \mathbb{C}$. Calculate $\theta(b), \theta(c)$ and $\theta(0)$.

    (b) Explain why $\theta$ is an indirect (orientation reversing) isometry.

    (c) Give a geometric interpretation of $\theta$.

    (d) Let $h = a + b + c$. Show that $t = (a - h)/(c - b)$ is purely imaginary by proving that $\overline{t} = -t$. By cyclically permuting $a, b$ and $c$ in this result, give a geometric interpretation to the point corresponding to the complex number $h$.

14. Consider a triangle $ABC$. Its altitudes meet at $H$, its *orthocentre*. Suppose that a point $P$ lies on the circumcircle of $ABC$. Consider the three points which are the reflections of $P$ in each of the three sides of $ABC$. Prove that these points are collinear, and are on a line passing through $H$. *A solution to the previous question will provide you with the complex number techniques which will allow you to prove this result.*