MA10209 Algebra 1A

Sheet 10 Problems v0: GCS

4-xii-11

The course website is http://people.bath.ac.uk/masgcs/diary.html

Hand in work to your tutor by 13:00, Monday December 12^{th}.

1. Let G be a group. Consider the map $s : G \longrightarrow G$ defined by $s(g) = g^2$. Prove that s is a homomorphism of groups if, and only if, G is abelian.

2. View the integers $\mathbb{Z}$ as a group under addition. Classify the subgroups of $\mathbb{Z}$ (i.e. describe them all in an organized way).

3. Let G be a group and suppose that X is a subset of G. Let $\langle X \rangle$ denote the intersection of all subgroups of G which contain the subset X. If $H = \langle X \rangle$, then we say that X *generates* H.

   (a) Prove that $\langle X \rangle$ is a subgroup of G.

   (b) Prove that G is a cyclic group if, and only if, there is a singleton set $T = \{t\}$ such that $G = \langle T \rangle$.

   (c) Suppose that $H \leq G$. Prove that there is $Y \subseteq G$ such that $H = \langle Y \rangle$.

4. Consider the set $\mathbb{Q}$ of rational numbers viewed as an additive group. We use the notation $+$ for the group operation, and 0 for the identity element. Suppose that $q_1, q_2 \in \mathbb{Q}$. Borrowing the notation of Problem 3, prove that there is $q_3 \in \mathbb{Q}$ such that $\langle \{q_1, q_2\} \rangle \subseteq \langle \{q_3\} \rangle$.

5. Let n be a natural number, and write $\Omega_n = \{1, 2, \ldots, n\}$. Let S_n denote the collection of all bijections $f : \Omega_n \longrightarrow \Omega_n$. Now S_n is a group where the operation is composition of maps.

   (a) Let H be the subset of S_5 consisting of all elements h of S_5 such that $h(1) = 1$. Is H a subgroup of S_5? Justify your answer.

   (b) Let K be the subset of S_6 consisting of all elements k of S_6 such that $k(1) = 2$. Is K a subgroup of S_6? Justify your answer.

   (c) Let L be the subset of S_7 consisting of all elements l of S_7 such that $l(i) - i$ is even for every $i \in \Omega_7$. Is L a subgroup of S_7? Justify your answer.

   (d) Let M be the subset of S_8 consisting of all elements m of S_8 such that $|\{i \mid i \in \Omega_8, m(i) = i\}|$ is even. Is M a subgroup of S_8? Justify your answer.

6. Suppose that G, H are groups and that f_1, f_2 are homomorphisms from G to H Let $K = \{g \mid g \in G, f_1(g) = f_2(g)\}$. Prove that $K \leq G$.

7. Let G be a group and suppose that K a subgroup of G, and H is a subgroup of K. The sets X, Y and Z are subsets of G.

   (a) Suppose that $K = \cup_{y \in Y} Hy$ and $G = \cup_{z \in Z} Kz$. Define $YZ = \{yz \mid y \in Y, z \in Z\}$. Prove that $G = \cup_{t \in YZ} Ht$.

   (b) Suppose that $G = \cup_{x \in X} Hx$. Show that $K = \cup_{x \in X \cap K} Hx$.

   (c) Suppose that $G = \cup_{z \in Z} Kz$. Prove that $G = \cup_{z \in Z} z^{-1}K$.

8. Suppose that A and B are subgroups of a group G. Let $AB = \{ab \mid a \in A, b \in B\}$. Define a map $\theta : A \times B \longrightarrow G$ by $\theta((a, b)) = ab$ for all $(a, b) \in A \times B$.

   (a) Suppose that $ab = a'b'$ for $a, a' \in A$ and $b, b' \in B$. Prove that $a^{-1}a' \in A \cap B$ so there is $c \in A \cap B$ such that $ac = a'$ and $c^{-1}b = b'$.

   (b) Suppose that $a \in A$ and $b \in B$ and $c \in A \cap B$. Prove that $ac \in A$, $c^{-1}b \in B$.

   (c) Suppose that G is finite. Deduce that all non-empty fibres of θ have the same size, and go on to conclude that
   $$|AB| = \frac{|A| \cdot |B|}{|A \cap B|}.$$

   (d) Suppose that U and V are subgroups of the finite group G, and $|U|, |V| > \sqrt{|G|}$. Prove that $U \cap V$ is not a trivial group.

9. (Harder but not too hard) Let G be a group, and suppose that X is a subset of G. Let W denote the set of elements of G which have the form
   $$x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_m^{\varepsilon_m}$$
   where $m \in \mathbb{N}$, $x_i \in X$ for each $i = 1, 2, \ldots, m$ and $\varepsilon_i \in \{1, -1\}$ for each $i = 1, 2, \ldots, m$. Borrowing the notation of Problem 3, prove that $W \cup \{1\} = \langle X \rangle$.

10. (Harder but not too hard) A group G is *finitely generated* if there is a finite subset X of G such that $G = \langle X \rangle$. Is the additive group of rational numbers a finitely generated group?