On the three diagonals of a cyclic quadrilateral

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Abstract. If the side lengths of a non-degenerate cyclic quadrilateral are given, but not necessarily in cyclic order, then three diagonal lengths arise in the resulting three cyclic quadrilaterals, just as three possible pairs of supplementary angles arise as opposite vertices, and where the diagonals intersect, in each of the three configurations. We obtain a formula for the sum of the lengths of the three diagonals minus the sum of the four sides which enables us to deduce the geometric inequality that the sum of the side lengths is less than the sum of the lengths of the three diagonals. We obtain another formula when these lengths are replaced by their squares, and this yields a similar inequality. A proof of both formulas is given which uses algebraic geometry, but which proceeds by analysis of degenerate situations. Two alternative proofs of the linear version of the inequality (which implies the quadratic version) are supplied which use trigonometry and Lagrange multipliers respectively. An unusual feature of these results is that they refer not to one configuration, but rather concern three possible configurations.

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1. Introduction

Let a, b, c and d be the lengths of the sides of a cyclic quadrilateral. If the cyclic order of the sides is not specified, then in general three pairwise non-congruent cyclic quadrilaterals arise. In degenerate situations, for example when $a = b$, then we do not get three non-congruent figures, and in such cases there may be various accidental co-incidences in the diagrams. We shall refrain from using the term "in general" in our discussions, but rather we deem it to be understood. The equations and inequalities that we discuss remain true in degenerate circumstances, by continuity.

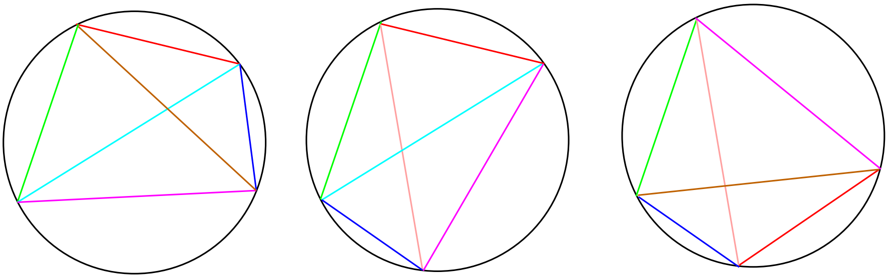


FIG. 1. The three possible cyclic quadrilaterals with given side lengths

The lengths a, b, c and d determine the common circumradius R and common area Δ of each of the three types of cyclic quadrilateral. In each quadrilateral, the three pairs of "angles and their supplements" specified by the two pairs of opposite vertices and the supplementary angles at the intersection of the diagonals will co-incide, but in each case the particular pair of supplementary angles defined by the diagonals will differ. The situation is illustrated in Figure 1. We have three cyclic quadrilaterals, where line segments of the same colour are congruent.

Theorem

Suppose that a non-degenerate cyclic quadrilateral has sides a, b, c and d, and the possible lengths of its diagonals are two of p, q and r. Denote by t the sum $a+b+c+d+p+q+r$ of all vertex to vertex distances which appear in the three diagrams. Then

$$(p+q+r)-(a+b+c+d)=\frac{abcd}{R^2 t}.$$

Corollary

(a) We have $a+b+c+d<p+q+r$ for non-degenerate cyclic quadrilaterals. Equality is achieved in two types of limit: when a side approaches 0, and when the sides approach non-zero limits but the circumradius tends to infinity (so the quadrilateral is becoming flat).

(b) In the course of the first proof we establish that

$$(p^2+q^2+r^2)-(a^2+b^2+c^2+d^2)=\frac{abcd}{R^2}$$

and so $a^2+b^2+c^2+d^2<p^2+q^2+r^2$.

Note that Corollary (b) is equivalent to the Theorem.

We remind the reader of the classical theorem of Ptolemy which states that the product of the (two) diagonal lengths in a cyclic quadrilateral is the sum of the products of the lengths of its opposite sides. A result which is less well-known is the formula $\frac{pqr}{4R}$ for the area of any one of the three cyclic quadrilaterals shown in Figure 1 [1]. To see why this is true, pick a quadrilateral and express its area as the sum of four small triangles, and express the area of each small triangle using "$\frac{1}{2}ab\sin C$" where the angle in question is one where the diagonals meet. Putting these together we obtain the formula $\frac{1}{2}pq\sin X$. Now move to one of the other two diagrams where X arises as a vertex angle, and eliminate $\sin X$ by means of the sine rule.

Proof. Since

$$(pq)(rp)/(qr) = p^2,\ (pq)(qr)/(rp) = q^2, \text{ and } (qr)(rp)/(pq) = r^2,$$

we can use three applications of Ptolemy's theorem to discover that $pq = ac + bd$, $qr = ad + bc$, and $rp = ab + cd$, and so are all quadratic polynomials in the lengths of the sides.

Notice that $(pqr)^2p^2 = (pq)^2(rp)^2$ is a homogeneous polynomial of degree 8 in the side lengths, as are $(pqr)^2q^2$ and $(pqr)^2r^2$.

Consider the expression

$$P = (pqr)^2((p+q+r)^2 - (a+b+c+d)^2).$$

Now by three applications of Ptolemy's theorem, P is

$$(pqr)^2(p^2+q^2+r^2-a^2-b^2-c^2-d^2),$$

a homogeneous symmetric polynomial of degree 8 in the sides. Now P vanishes when $a = 0$. Write $s = (a+b+c+d)/2$ for the semiperimeter of a quadrilateral. Then P approaches 0 as the sides approach non-zero limits but the circumradius tends to infinity (so the quadrilateral is becoming flat), and this happens as a side approaches s. Therefore P is a real multiple of $abcd(s-a)(s-b)(s-c)(s-d)$, and by considering a square we see that $P = 16abcd(s-a)(s-b)(s-c)(s-d)$.

Now $(s-a)(s-b)(s-c)(s-d)$ and $(pqr)^2/16R^2$ are both Δ^2, so

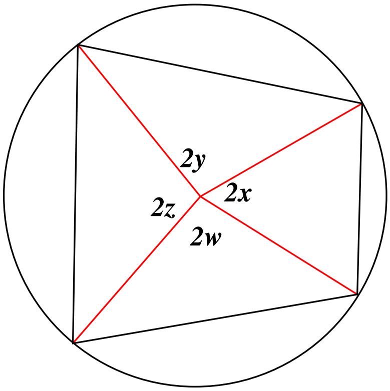
$$p^2+q^2+r^2-a^2-b^2-c^2-d^2 = (p+q+r)^2-(a+b+c+d)^2 = \frac{abcd}{R^2}$$

and we may divide by t to complete the proof. □

2. Alternative proofs of the corollary

Our original ambition was to establish part (a) of the Corollary (from which part (b) is immediate). We include two different arguments which establish this without recourse to the Theorem. The first argument is trigonometric, and the second uses Lagrange multipliers.

Proof. Let $2x, 2y, 2z$ and $2w$ be the angles at the centre which are subtended by the sides, and so the sides have lengths $2R\sin x$, $2R\sin y$, $2R\sin z$ and $2R\sin(x+y+z)$ respectively, while the diagonals have lengths $2R\sin(x+y)$, $2R\sin(y+z)$ and



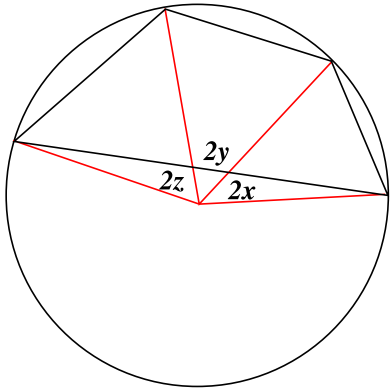


FIG. 2. The two possible configurations

$2R\sin(z+x)$. This is so, since either $w = \pi-(x+y+z)$ (when the quadrilateral contains its circumcentre), or $w = x+y+z$ (when the quadrilateral does not contain its circumcentre), as illustrated in Figure 2.

Let us construct, on the domain $\mathscr{D} = \{(x, y, z) \in \mathbb{R}^3_+ \mid x+y+z \le \pi\}$, the function $L(x, y, z)$ defined by

$$L = \sin x + \sin y + \sin z + \sin(x+y+z) - \sin(x+y) - \sin(y+z) - \sin(z+x).$$

This expression for $L(x, y, z)$ factorizes as

$$L(x, y, z) = -8\sin\frac{x}{2}\sin\frac{y}{2}\sin\frac{z}{2}\cos\frac{x+y+z}{2},$$

and so is negative in the interior of the domain $\mathscr{D}$, and takes the value zero precisely when any of x, y, z is 0, or when $x+y+z = \pi$, i.e. on the whole boundary $\partial\mathscr{D}$ of the domain. This factorization obtained can be found by complex numbers.

Taking a product over x, y and z we have

$$\prod\left(e^{ix}-1\right) = e^{i(x+y+z)} - \sum e^{i(x+y)} + \sum e^{ix} - 1,$$

which has imaginary part $L(x, y, z)$. On the other hand, since $e^{iu}-1 = \left(2i\sin\frac{u}{2}\right)e^{iu/2}$, we have

$$\prod\left(e^{ix}-1\right) = (2i)^3\left(\prod\sin\frac{x}{2}\right)\left(\prod e^{ix/2}\right) = -8i\left(\prod\sin\frac{x}{2}\right)e^{i(x+y+z)/2},$$

with (negative) imaginary part $-8\sin\frac{x}{2}\sin\frac{y}{2}\sin\frac{z}{2}\cos\frac{x+y+z}{2}$. □

It is also possible to prove this result by means of the method of Lagrange Multipliers.

Proof. Let us construct the (same) general Lagrange function $L(x,y,z)$ on $\mathbb{R}^3$

$$L = \sin x + \sin y + \sin z + \sin(x+y+z) - \sin(x+y) - \sin(y+z) - \sin(z+x).$$

Now
$$\begin{cases} \dfrac{\partial}{\partial x}L(x,y,z) &= \cos x + \cos(x+y+z) - \cos(x+y) - \cos(z+x) \\ \dfrac{\partial}{\partial y}L(x,y,z) &= \cos y + \cos(x+y+z) - \cos(x+y) - \cos(y+z) \\ \dfrac{\partial}{\partial z}L(x,y,z) &= \cos z + \cos(x+y+z) - \cos(y+z) - \cos(z+x) \end{cases}$$

are set to zero, in order to find the critical points of L.

Subtract in pairs, to get
$$\begin{cases} \sin\dfrac{x-y}{2}\left(\sin\dfrac{x+y}{2} - \sin\dfrac{x+y+2z}{2}\right) &= 0 \\ \sin\dfrac{y-z}{2}\left(\sin\dfrac{y+z}{2} - \sin\dfrac{2x+y+z}{2}\right) &= 0 \\ \sin\dfrac{z-x}{2}\left(\sin\dfrac{z+x}{2} - \sin\dfrac{x+2y+z}{2}\right) &= 0 \end{cases}.$$

Now, $x+y+z \equiv \pi \pmod{2\pi}$ implies $\sin\dfrac{x+y}{2} - \sin\dfrac{x+y+2z}{2} = 0$ and all similar others, and so all three of the above relations are satisfied, but then $L(x,y,z) = 0$. Otherwise, if any of x,y,z is $\equiv 0 \pmod{2\pi}$ then again $L(x,y,z) = 0$. (Critical points are when any two of x,y,z are $\equiv 0 \pmod{2\pi}$). If we have none of the above, then we need $x \equiv y \equiv z \equiv \alpha \pmod{2\pi}$, with, to avoid the previous cases, $\alpha \not\equiv 0, \pi, \pm\pi/3 \pmod{2\pi}$. Then

$$L(\alpha,\alpha,\alpha) = 3\sin\alpha + \sin 3\alpha - 3\sin 2\alpha = 2\sin\alpha(1-\cos\alpha)(1-2\cos\alpha).$$

The roots are $\alpha \equiv 0, \pi, \pm\pi/3 \pmod{2\pi}$, precisely those values already considered. The function is periodic with principal period 2π. On $(0,2\pi)$ the function is negative in $(0,\pi/3) \cup (\pi, 2\pi-\pi/3)$ and non-negative elsewhere, with global maximum $3+2\sqrt{2}$ at $\alpha = 2\arctan(1+\sqrt{2})$ and global minimum $-3-2\sqrt{2}$ at $\alpha = 2\pi - 2\arctan(1+\sqrt{2})$. The local maximum is $3-2\sqrt{2}$ at $\alpha = 2\pi - \pi/4$ and the local minimum is $-3+2\sqrt{2}$ at $\alpha = \pi/4$.

Now notice that the zeros of L determine the boundary of the domain

$$\mathscr{D} = \{(x,y,z) \in \mathbb{R}^3_+ \mid x+y+z \le \pi\},$$

of interest to us for the geometric interpretation. The values of L are negative in the interior of the domain (and zero on the boundary $\partial\mathscr{D}$, which in fact deals with the degenerate cases, when at least one angle of the x,y,z,t is 0, and the quadrilateral degenerates to a triangle, a segment, or a point). (Also, the least value taken is $-3+2\sqrt{2}$ when the quadrilateral is a square). □

3. Concluding Remarks

Let $ABCD$ be a cyclic quadrilateral with circumradius R. From the triangle inequality, we have

$$AC + BD < AB + BC + CD + DA < 2(AC + BD) \leq 8R.$$

The first author sought to tighten this result and obtained

$$AB + BC + CD + DA < AC + BD + 2R$$

with 2 the best possible constant. The proof was by the method of Lagrange multipliers. The advantage of this formulation is that it refers only to explicit features of a particular cyclic quadrilateral. However, the disadvantage is that it hides the three-fold character of a cyclic quadrilateral, and by being more particular, it seems that it becomes more difficult to establish the cruder inequality directly.

A referee kindly pointed out another way to prove the Theorem, by means of the three forms of Ptolemy's theorem, Parameshvara's formula for the circumradius of a cyclic quadrilateral, and a Computer Algebra system to factor $p^2 + q^2 + r^2 - a^2 - b^2 - c^2 - d^2$.

References

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