Rings with periodic groups of units

Jan Krempa

(Notes for a talk on Burnside day)

During this talk R stands for an associative, unital ring, $\mathrm{U}(R)$ denotes the unit group of the ring R, and R_U – the subring of R generated by $\mathrm{U}(R)$.

William Burnside was probably the first person to prove an essential result on periodic subgroups of units of noncommutative rings. With further generalization this result can be stated as follows:

Theorem 1 (Burnside & Schur) *Every periodic linear group over a field is locally finite.*

About 50 years later the above classical theorem was extended to the following :

Theorem 2 (Shirshov & Procesi) *Let R be a ring and let G be a periodic subgroup of $U(R)$. If R satisfies a nontrivial polynomial identity (is a PI-ring) then G is locally finite.*

Remark. Any group G is a subgroup of units, for example of the integral group ring ZG. Hence, by the negative answer to the general Burnside problem, not every periodic subgroup of units of a ring has to be locally finite.

Further, if not stated otherwise, all rings have *torsion free additive groups*, if $P \subset R$ is a minimal prime ideal then the factor ring R/P will be called a *main factor of R*. Here orders are Z - orders.

For groups we use the following notation:

- C_n - The cyclic group of order n;
- Q_8 - The quaternion group of order 8;
- $DC_{12} = \langle a, b \mid a^3 = b^2 = (ab)^2 = 1\rangle$, $(|DC_{12}| = 12)$;
- $BT_{24} = \langle a, b \mid a^3 = b^3 = (ab)^2 = 1\rangle$, $(|BT_{24}| = 24)$;

- G stands for a nontrivial group.

Lemma 1 (Hirsch & Zassenhaus) *Let R be a prime order such that $R = R_U$. Then:*

1. $\mathrm{U}(R) \cong C_2$ *if and only if* $R = Z$;
2. $\mathrm{U}(R) \cong C_4$ *if and only if* $R = Z[i]$;
3. $\mathrm{U}(R) \cong C_6$ *if and only if* $R = Z[\zeta_3]$;
4. $\mathrm{U}(R) \cong Q_8$ *if and only if* $R = Z[i, j]$;
5. $\mathrm{U}(R) \cong DC_{12}$ *if and only if* $R = Z[\zeta_3, j]$;
6. $\mathrm{U}(R) \cong BT_{24}$ *if and only if* $R = Z[i, j, l]$, *where* $l = \frac{1+i+j+k}{2}$.

Groups and rings listed in the above lemma will be called *acceptable.*

Theorem 3 *Let R be an order such that $R = R_U$. Then $\mathrm{U}(R)$ is periodic if and only if R is a finite subdirect product of acceptable rings.*
Hence $\mathrm{U}(R)$ is periodic if and only if it is finite.

Without assumption on R being an order we have

Theorem 4 *Let R be such that $R = R_U$. Then the following conditions are equivalent:*

1. *The group $\mathrm{U}(R)$ is periodic;*
2. *Every finitely generated subring of R is an order with a finite group of units;*
3. *R is semiprime and every its main factor is acceptable;*
4. *$\mathrm{U}(R)$ is a subdirect product of acceptable groups;*
5. *The group $\mathrm{U}(R)$ is locally finite.*

Corollary 2 *Let $R = R_U$. If $\mathrm{U}(R)$ is periodic then R satisfies the standard identity of degree* 4, *and $\mathrm{U}(R)$ is a soluble group of class at most* 3 *and of exponent dividing* 12.

Theorem 5 *Let R be such that $R = R_U$, and let G be any group.*

1. *If G is an elementary Abelian 2-group then* $\mathrm{U}(RG)$ *is periodic if and only if* $\mathrm{U}(R)$ *is periodic;*

2. *If G is Abelian of exponent 3 or 6 then* $\mathrm{U}(RG)$ *is periodic if and only if R is semiprime and any of its main factors is isomorphic either to Z or to* $Z[\zeta_3]$;

3. *If G is Abelian of exponent 4 then* $\mathrm{U}(RG)$ *is periodic if and only if R is semiprime and any of its main factors is isomorphic either to Z or to* $Z[i]$;

4. *If G is a Hamiltonian 2-group then* $\mathrm{U}(RG)$ *is periodic if and only if R is semiprime and any of its main factors is isomprphic to Z.*

5. *In any other case* $\mathrm{U}(RG)$ *is not periodic.*

Theorem 6 *Let R be such that $R = R_U$, and let G be a periodic group. If R has no proper idempotents then* $\mathrm{U}(RG)$ *is periodic if and only if* $\mathrm{U}(R)$ *is periodic and* $\mathrm{U}(RG) = \mathrm{U}(R) \cdot G$.

Corollary 3 (Higman) *For a group G the following conditions are equivalent:*

1. $\mathrm{U}(ZG)$ *is periodic;*

2. *Either G is Abelian of exponent $2, 3, 4$ or 6, or G is a Hamiltonian 2-group;*

3. *G is periodic and* $\mathrm{U}(ZG) = \mathrm{U}(Z) \cdot G$.

Corollary 4 *Let $R = Z[i]$. For a group G the following conditions are equivalent:*

1. $\mathrm{U}(RG)$ *is periodic;*

2. *G is Abelian of exponent either 2 or 4;*

3. *G is periodic and* $\mathrm{U}(RG) = \mathrm{U}(R) \cdot G$.

Corollary 5 *Let $R = Z[\zeta_3]$. For a group G the following conditions are equivalent:*

1. $\mathrm{U}(RG)$ *is periodic;*

2. *G is Abelian of exponent either 2 or 3 or 6;*

3. G *is periodic and* $\mathrm{U}(RG) = \mathrm{U}(R) \cdot G$*.*

Corollary 6 *Let* $R = Z[i,j]$*,* $Z[\zeta_3, j]$*, or* $Z[i,j,l]$*, where* $l = \frac{1+i+j+k}{2}$*. For a group* G *the following conditions are equivalent:*

1. $\mathrm{U}(RG)$ *is periodic;*
2. G *is elementary Abelian of exponent* 2*;*
3. G *is periodic and* $\mathrm{U}(RG) = \mathrm{U}(R) \cdot G$*.*

Corollary 7 *If* R *is a prime ring with* $R = R_U$ *and* G *is a group such that* $\mathrm{U}(RG)$ *is periodic, then* R *and* G *have to satisfy conditions from one of the above corollaries.*

Some interesting results about periodic subgroups of group rings were proved by A.A. Bovdi, S.P. Coelho and C. Polcino Milies. Now let us consider some other classes of rings.

Theorem 7 (Wedderburn) *Let* D *be a division algebra. Then* $\mathrm{U}(D)$ *is periodic if and only if* D *is a field of positive characteristic, which is algebraic over its prime subfield.*

From now F represents a field of characteristic $p > 0$ which is algebraic over its prime subfield, and $\mathrm{J}(R)$ denotes the Jacobson radical of a ring R.

Example 1. Let K, L be nontrivial field extensions of F and let A be the free coproduct of them over F. Then, by results of G.M. Bergman, A is a domain, $A = A_U$, and $\mathrm{U}(A)$ contains many periodic elements. But $\mathrm{U}(A)$ is not periodic, and A is not algebraic over F.

Question. *Let an* F*-algebra* D *be a domain such that* $D = D_U$ *and* $\mathrm{U}(D)$ *is periodic. Is* D *locally finite, or at least algebraic over* F*?*
From the theorem of Shirshov and Procesi we know that the answer is “Yes” when R is a PI-ring.

Now let us consider algebras, which are not domains.

Theorem 8 (Golod) *There exists an* F*-algebra* A *such that* $A = F + \mathrm{J}(A)$*,* $\mathrm{U}(A)$ *is periodic, but not locally finite.*

Theorem 9 (Amitsur) *There exists a semisimple, even left and right primitive, algebraic* F*-algebra* B *such that* $B = B_U$ *and* $\mathrm{U}(B)$ *is periodic, but* $\mathrm{U}(B)$ *is not locally finite.*

Following G.M. Bergman an F-algebra A will be called *matrix algebraic* if for any $n \in N$ the matrix algebra $\mathrm{M}_n(A)$ is algebraic over F.

Theorem 10 *Let A be a F-algebra. Then the following conditions are equivalent:*

1. *The group $\mathrm{GL}_n(A)$ is periodic for any $n \in N$;*
2. *The group $\mathrm{GL}_n(A)$ does not contain a noncyclic free subgroup for any $n \in N$;*
3. *A is matrix algebraic.*

Lemma 8 *Let $R = A[x]$ where A is any ring. Then the group $\mathrm{GL}_2(R)$ contains a nontrivial free subgroup.*

Example 2. Let $A = Z[x, y, z, u]$ be the integral polynomial ring in noncommutative indeterminates and let $J \subset A$ be an ideal generated by elements $xz - 1$, $zx - 1$, $yu - 1$, $uy - 1$. Let us put $R = A/pJ$ for a prime number $p \in N$. Because $pJ = pA \cap J$, one can calculate that $\mathrm{U}(R) = C_2$, torsion part, say T of R^+ is equal to J/pJ and $R/T = A/J$ is isomorphic to the group ring ZF where F is the free group of rank 2. In particular $\mathrm{U}(R/T)$ is nonperiodic and satisfies no nontrivial group identity.

References

[1] G.M. Bergman, *Radicals, tensor products, and algebraicity,* Israel Math. Conf. Proc. vol. 1, The Weizmann Science Press of Israel (1989), 150-192.

[2] S.P. Coelho & C. Polcino Milies, *Group rings whose torsion units form a subgroup,* Proc. Edinburgh Math. Soc. 37(1994), 201-205.

[3] L. Fuchs, *"Infinite Abelian groups",* vol. 2, Academic Press, New York 1973.

[4] K.A. Hirsch, & H. Zassenhaus, *Finite automorphism groups of torsion-free groups,* J. London Math. Soc. 41(1966), 545-549.

[5] E. Jespers, M.M. Parmenter, & P.F. Smith, *Revisiting a theorem of Higman,* in: *"Groups '93 Galway/St Andrews",* vol 1, London Math. Soc. Lecture Note 211, Campbell C.M. Hurley T.C. Robertson E.F. Tobin S.J. & Ward J.J. (eds), Cambridge University Press, Cambridge 1995. pp 269-273.

[6] G. Karpilovsky, *"Unit groups of group rings",* Longman, Essex 1989.

[7] J. Krempa, *On finite generation of unit group for group rings,* in: "Groups '93 Galway/St Andrews", vol. 2, London Math. Soc. Lecture Note 212, Cambridge University Press, Cambridge 1995, pp 352-367.

[8] —, *Rings with periodic unit groups.* in: "Abelian groups and modules", A. Facchini & C. Menini (eds.), Kluwer Academic Publishers, Dordrecht 1995, pp 313-321.

[9] D.S. Passman, *"The algebraic structure of group rings",* Wiley-Interscience Publications, New York 1977.

[10] S.K. Sehgal, *"Topics in group rings",* Marcel Dekker Inc., New York 1978.

[11] —, *"Units in integral group rings",* Longman, Essex 1993.