MA10209 Algebra 1A

Sheet 6 Problems : GCS

5-xi-2018

Hand in work to your tutor at the time specified by your tutor. The latest possible hand in time will be 17:15, Monday Nov 12th.
Course website `http://people.bath.ac.uk/masgcs/diary.html`

1. How many integers in the range $0 \leq i \leq 2014$ are coprime to 2015?

2. Suppose that m is an odd natural number. Prove that there is a natural number n such that m divides $2^n - 1$.

3. Find all integers x such that $x \equiv 3 \bmod 7$ and $x \equiv 4 \bmod 9$.

4. Find all integers y such that 9 divides $2y + 1$ and 11 divides $3y + 6$.

5. Find the smallest positive integer z such that $z \equiv 10 \bmod 11$, $z \equiv 12 \bmod 13$, $z \equiv 17 \bmod 18$. *Hint: this is much easier than it looks.*

6. Suppose that $p > 3$ is a prime number. Prove that $2^{p-2} + 3^{p-2} + 6^{p-2} - 1$ is a multiple of p.

7. Show that there are 1000 consecutive positive integers, each of which is divisible by at least 1000 different prime numbers.

8. Suppose that $m, n \in \mathbb{N}$. Consider the map $\pi_{mn} : \mathbb{Z}_{mn} \longrightarrow \mathbb{Z}_m \times \mathbb{Z}_n$ defined by $[x]_{mn} \mapsto ([x]_m, [x]_n)$ for each $x \in \mathbb{Z}$, where $[x]_k$ denotes the equivalence class of x under the relation $\sim_k$. Determine $|\mathrm{Im}\ \pi_{mn}|$. *Note that if m and n are coprime, then the Chinese Remainder Theorem applies and the map π is surjective. In that case, the size of the image is therefore mn. This question therefore requires an investigation of how the CRT fails when m and n are not coprime.*

9. Let d be a positive integer. A d-arithmetic set is defined to be a set of the form $\{a + md \ \mid m = 0, 1, 2, \ldots\}$ for some positive integer a. Suppose that $N > 1$ is a positive integer and that we have a p-arithmetic set S_p for each prime number $p \leq N$. Show that there are $2N + 1$ consecutive positive integers, all except two of which are in the union S of our sets S_p. *Hint: CRT & Eratosthenes*

10. (Challenge!) A mathematical tree (i.e. a vertical unit interval) grows at each point of an infinite plane with integral co-ordinates except for the origin $(0,0)$ where an observer, of height 1, stands. Many trees are visible, including those at $(1,0),(7,8)$ and $(45,-7)$. Other trees are invisible, because the view of them from the origin is obstructed by other trees. For example, the view of the tree at $(-14,91)$ is obstructed by the tree at $(-2,13)$.

    Show that it is possible for a *Tunguska event* of diameter 10^{10} to happen, yet be unknown to the observer. In other words, show that there is a circle in the plane of diameter 10^{10} which has only invisible trees in its interior.