MA10209 Algebra 1A

Solutions to vacation Problems: GCS

The course website is http://people.bath.ac.uk/masgcs/diary.html

This work is not to be handed in.

1. Search the internet on terms such as *Platonic solids* and *mathematical origami* to find many sites which will teach you how to make Platonic solids from paper. Paper models of Platonic solids make excellent baubles when decorating a tree.
   Solution I hope it went well.

2. Think deep thoughts about the group of orientation preserving rigid symmetries of a cube. In lectures you were told that this group was a copy of S_4, and that the natural set of four things being permuted is the collection of grand (or great) diagonals. Make a cube (from hard cheese or raw potato) and indicate the grand diagonals by means of cocktail sticks, labelled from 1 to 4. Is it really true that there are elements of this group corresponding to each possible permutation of these four things? Is it also true that if two orientation preserving rigid symmetries of a cube permute the cocktail sticks in the same way, then they are the same element?
   Solution Label the vertices round a face 1,2,3,4 in anticlockwise order. Transfer these labels to the grand diagonals through those vertices, and it will be helpful to also put those labels on these grand diagonals where they emerge from the other side of the cube.

   Now describe some rigid motions in terms of the way the grand diagonals are permuted. Rotation the cube through $2\pi/3$ about the grand diagonal labelled 1 gives the permutation (243) one way, and (234) the other. These are both even permutations of the grand diagonals. Using the other three grand diagonal in the same way, we find eight event permutations in total. The group H that they generate (in the sense of Sheet 10) will be a subgroup of A_4 which has order 12. By Lagrange's theorem, $H = A_4$.

   Now consider rotating the cube through $\pi/2$ about an axis through the centre of, and perpendicular to, the face that was first mentioned. This permutes the grand diagonals by (1234). This is an odd permutation. The group G generated by the 13 elements we have so far discovered must be a subgroup of S_4 (of order 24), and so by Lagrange's theorem, $G = S_4$.

   The group of R of rigid symmetries of a cube has order 24 because you can put any one of six faces on the bottom, and this bottom face can be kept on the bottom by four rotations. We have a map (a group homomorphism in fact) $\psi : R \longrightarrow S_4$ defined for each $r \in R$ by the permutation of the grand diagonals which r induces. Our argument above shows that ψ is surjective. However, both R and S_4 have order 24 so ψ is bijective, and so is an isomorphism. Therefore the two questions both have affirmative answers.

3. Consider the matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}$ and, as on sheet 8, use A to define a linear map $f_A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$. Now define a new map $\widehat{f} : \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2$, by restricting the domain of f_A. Put another way, if $z \in \mathbb{Z}^2$ we define $\widehat{f}(z)$ to be $f_A(z)$.

   (a) Prove that $\widehat{f}(z) : \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2$ is a homomorphism of groups (using co-ordinatewise addition).
   Solution Suppose that $u, v \in \mathbb{Z}^2$, so $u = (u_1, u_2)$ and $v = (v_1, v_2)$ for integers u_1, u_2, v_1 and v_2. Now

$\widehat{f}(u+v) = (k, l)$ where

$$\begin{pmatrix} k \\ l \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

Therefore $\widehat{f}(u+v) = \widehat{f}(u) + \widehat{f}(v)$ for all u and v, so $\widehat{f}$ is a homomorphism of groups.

(b) Calculate $\widehat{f}((1,0))$ and $\widehat{f}((0,1))$.
Solution $f((1,0)) = (2,1)$ and $\widehat{f}((0,1)) = (1,4)$.

(c) Make festive decorative paper as follows. Use a white piece of paper equipped with a Cartesian co-ordinates. Colour points P with co-ordinates $(x,y) \in \mathbb{Z}^2$ red if $(x,y) \in \operatorname{Im} \widehat{f}$. Colour points Q with co-ordinates $(u,v) \in \mathbb{Z}^2$ blue if $(u,v) \notin \operatorname{Im} \widehat{f}$.
Solution I hope that it went well.

(d) Observe that $\widehat{f}$ is not surjective, even though f_A is bijective.
Solution There are some blue spots, so $\widehat{f}$ is not surjective.

(e) Is the determinant of A somehow visible in the decorations?
Solution The determinant of A is 7. This is visible in the diagram because $\mathbb{Z}^2$ is the union of $\operatorname{Im} \widehat{f}$ and exactly 6 other cosets of $\operatorname{Im} \widehat{f}$ in $\mathbb{Z}^2$. The elements of $\operatorname{Im} \widehat{f}$ are red, but the elements of the remaining $6 = 7 - 1$ cosets are blue.

Consider a convex subset of the plane, and consider the proportion of grid points inside the square which are coloured red. Now enlarge the region. As the area of the region goes to ∞, the proportion of red points amongst the grid points interior to the region approaches $1/7 = (\det \mathrm{A})^{-1}$.

4. Take a regular heptagon (use the vertices of a 50 pence coin) and label its vertices anticlockwise with the numbers 1 through to 7. There are 14 rigid symmetries of the set of vertices, each of which can be regarded as an element of S_7. For example, rotation about the centre through $2\pi/7$ is denoted (1,2,3,4,5,6,7). It is also $(2,3,4,5,6,7,1)$ but that is just a different name for the same map. Alternatively, you could turn the coin over about an axis through 1 and its centre. This rigid motion is denoted $(2,7)(3,6)(4,5)$. Write down all 14 elements of this group in cycle notation.
Solution The rotations are id, $(1,2,3,4,5,6,7)$, $(1,3,5,7,2,4,6)$, $(1,4,7,3,6,2,5)$, $(1,5,2,6,3,7,4)$, $(1,6,4,2,7,5,3)$ and $(1,7,6,5,4,3,2,1)$. The reflections are $(2,7)(3,6)(4,5)$, $(1,3)(7,4)(6,5)$, $(2,4)(1,5)(7,6)$, $(3,5)(2,6)(1,7)$, $(4,6)(3,7)(2,1)$ and $(5,7)(4,1)(3,2)$.

5. The group in Problem 4 is called the *dihedral group* of order 14. We will call it D_{14} (but some call it D_7). Given any $n \geq 3$, there is a corresponding group of rigid symmetries of a regular n-gon (an equilateral triangle is a regular 3-gon and a square is a regular 4-gon). Using a numbering convention similar to that in Problem 4, write down the eight elements of D_8 in cycle notation.
Solution This is the group of rigid symmetries (isometries) of a square (a regular 4-gon). The rotations are id, $(1,2,3,4)$, $(1,3)(2,4)$, $(1,4,3,2)$ The reflections are $(1,2)(3,4)$, $(1,4)(2,3)$, $(1,3)$ and $(2,4)$.

6. Consider a rectangle which is not a square.

(a) Use the an anticlockwise numbering system for its vertices to write down the elements of V, its group of rigid symmetries. There should be four of them (you are allowed to turn the rectangle over).
Solution The elements are id, $(1,3)(2,4)$, $(1,4)(2,3)$ and $(1,2)(3,4)$.

(b) Write down the multiplication table of V.
Solution Let the elements just mentioned be x, y and z respectively. Let e denote the identity element.

The multiplication table is

	e	x	y	z
e	e	x	y	z
x	x	e	z	y
y	y	z	e	x
z	z	y	x	e

.

7. Find all the subgroups of D_8. You may find that Lagrange's Theorem is very helpful when doing this analysis.
Solution Label the vertices of the square $1, 2, 3, 4$ anticlockwise. The eight elements of D_8 are id, $r = (1, 2, 3, 4)$, $r^2 = (1, 3)(2, 4)$, $r^3 = (1, 4, 3, 2)$, $s = (1, 3)$, $sr = (1, 2)(3, 4) = r^3 s$, $sr^2 = (2, 4) = r^2 s$ and $sr^3 = (1, 4)(2, 3) = rs$. There are two obvious subgroups $R = \{\text{id}, r, r^2, r^3\}$ and $S = \{\text{id}, s\}$. Notice that $D_8 = R \cup Rs = sR \cup R$ and $D_8 = S \cup Sr \cup Sr^2 \cup Sr^3 = S \cup rS \cup r^2 S \cup r^3 S$. Notice that $sr^2 = s^2 r$ so $V = \{\text{id}, s, r^2, r^2 s\}$ is an abelian subgroup of size 4.

All that was by way of preamble, just setting up a decent description of D_8. By the theorem of Lagrange, all subgroups must have order 1, 2, 4 or 8. Now the trivial group is the only subgroup of size 2. A group of size 2 must consist of the identity together with an element of order 2. Conversely, any element of order 2 will give rise to a subgroup of order 2. The subgroups of order 2 are therefore Now r and r^3 have order 4, but allfive other non-identity elements of D_8 have order 2, and so there are five subgroups of order 2.

A group of size 4 must either be the set of powers of an element of order 4, or it must consist of the identity together with three elements, each of which has order 2. There is a group of the cyclic type: $\{\text{id}, r, r^2, r^3\} = \langle r \rangle$. Identifying subgroups of order 4 of the other type is not so easy. However, V is certainly an example.

Suppose that H is any group of order 4 in which the non-identity elements are all of order 2. Suppose that $H = \{\text{id}, a, b, c\}$. Notice that $ab \neq \text{id}$ because $a^2 = \text{id}$. Also $ab \neq a$ because then $b = \text{id}$. Similarly $ab \neq b$. Therefore $ab = c$. Similarly $ba = c$. Thus the non-identity elements of H commute. However the identity element commutes with all elements of H, so H is abelian.

Thus these groups we are looking for must be abelian. There are four elements of order 2 other than r^2. They are $s, rs, r^2 s$ and $r^3 s$. By inspection, each of them commutes with r^2 and itself, but does not commute with the other three elements of order 2. Therefore the only candidates for non-cyclic subgroups of size 4 are $V = \{\text{id}, r^2, s, r^2 s\}$ and $V_1 = \{\text{id}, r^2, rs, r^3 s\}$. In fact V_1 is a subgroup, as you may verify directly.

Finally D_8 is a subgroup of itself, and this is the only possible subgroup of order 8.

8. Let G be a group. For each $x \in G$, define a map $\tau_x : G \longrightarrow G$ by $\tau_x(g) = xgx^{-1}$.

   (a) Prove that each τ_x is a homomorphism.
   Solution Suppose that $x, g, h \in G$, then $\tau_x(gh) = xghx^{-1} = xgx^{-1}xhx^{-1} = \tau_x(g)\tau_x(h)$ so $\tau_x : G \longrightarrow G$ is a homomorphism.

   (b) Prove that each τ_x is a bijective map.
   Solution Observe that $(\tau_x \circ \tau_{x^{-1}})(g) = xx^{-1}gx^{-1}x = g$ for every $g \in G$. Therefore $\tau_x \circ \tau_{x^{-1}} = \text{Id}_G$ and replacing x by x^{-1} we have $\tau_{x^{-1}} \circ \tau_x = \text{Id}_G$. Here Id_G denotes the identity map from G to G, and not the identity element of the group G. Now each map τ_x has a two-sided inverse, and is therefore bijective.

   (c) Conclude that each $\tau_x \in \text{Aut }(G)$, i.e. that each τ_x is an automorphism of G.
   Solution We have proved that each $\tau_x : G \longrightarrow G$ is a bijective homomorphism, and hence an automorphism of G.

   (d) Consider the map $\tau : G \longrightarrow \text{Aut }(G)$ defined by $x \mapsto \tau_x$ for each $x \in G$. Prove that τ is a homomorphism.
   Solution Suppose that $g, h \in G$, then $\tau(gh) = \tau_{gh} : G \longrightarrow G$. Now for all $x \in G$ we have $(\tau(g) \circ \tau(h))(x) = (\tau_g \circ \tau_h)(x) = \tau_g(\tau_h(x)) = \tau_g(hxh^{-1}) = ghxh^{-}g^{-1} = (gh)x(gh)^{-1} = \tau(gh)(x)$. Since $\tau(gh), \tau(g)$ and $\tau(h)$ have the same domain, the same codomain, and act the same way we have $\tau(gh) = \tau(g) \circ \tau(h)$ so τ is a group homomorphism.

(e) Prove that Ker $\tau = \{y \mid y \in G, yz = zy \text{ for all } z \in G\}$.
Solution Ker $\tau = \{y \mid y \in G, \tau_y(z) = z \text{ for all } z \in G\}$. Therefore Ker $\tau = \{y \mid y \in G, yzy^{-1}z \text{ for all } z \in G\} = \{y \mid y \in G, yz = zy \text{ for all } z \in G\}$

9. Let $S_5 = \mathrm{Sym}(5)$ denote the *symmetric* group on $\Omega_5 = \{1, 2, 3, 4, 5\}$. Thus S_5 consists of the all the permutations of Ω_5. Give an example of an element of S_5 of each possible cycle shape, and work out how many elements there are of each cycle shape (making sure that your answers add to $120 = |S_5|$.
Solution The identity element is in S_5. This can be viewed as a product of disjoint 1-cycles. We will classify the elements of S_5 in terms of the size of their *support*, i.e. the number of elements of Ω_5 which they move. There are no group elements with support of size 1. The elements with support of size 2 are the transpositions such as $(1, 2)$. There are $\binom{5}{2} = 10$ of them. The elements with support of size 3 are all 3-cycles such as $(1, 2, 3)$. There are $2\binom{5}{3} = 20$ of these. The elements with support of size 4 are either 4-cycles such as $(1, 2, 3, 4)$, and there are $5(3!) = 30$ of those, or products of two disjoint transpositions such as $(1, 2)(3, 4)$, and there are $5 \cdot 3 = 15$ of them. The elements with support of size 5 are either 5-cycles such as $(1, 2, 3, 4, 5)$, and there are $4! = 24$ of those, or products of a 2-cycle with a disjoint 3-cycle such as $(1, 2)(3, 4, 5)$. There are $2\binom{5}{2} = 20$ of these.

Now $1 + 10 + 20 + 30 + 15 + 24 + 20 = 120 = 5!$ as required.

10. The group A_n studied in lectures is often called the *alternating group*. Let $A_5 = \mathrm{Alt}(5)$ denote the alternating group on $\Omega_5 = \{1, 2, 3, 4, 5\}$. Thus A_5 consists of the *even* permutations of Ω_5. Give an example of an element of A_5 of each possible cycle shape, and work out how many elements there are of each cycle shape (making sure that your answers add to $60 = |A_5|$.
Solution The even elements of S_5 are the identity, and those of support of size 3 which are all 3-cycles such as $(1, 2, 3)$. There are $2\binom{5}{3} = 20$ of these. There are also those of support of size 4 which are the products of two disjoint transpositions such as $(1, 2)(3, 4)$, and there are $5 \cdot 3 = 15$ of them. The elements with support of size 5 are either 5-cycles such as $(1, 2, 3, 4, 5)$, and there are $4! = 24$ of those.

Now $1 + 20 + 15 + 24 = 60 = (5!)/2$ as required.

11. We use a bar to denote complex conjugation. Suppose that α and β are complex numbers.
Solution Let $\alpha = u + iv$ and $\beta = x + iy$ where $u, v, x, y \in \mathbb{R}$.

(a) Prove that $\overline{\alpha + \beta} = \overline{\alpha} + \overline{\beta}$.
Solution $\overline{\alpha + \beta} = \overline{(u + x) + i(v + y)} = (u + x) - i(v + y) = (u - iv) + (x - iy) = \overline{\alpha} + \overline{\beta}$.

(b) Prove that $\overline{\alpha - \beta} = \overline{\alpha} - \overline{\beta}$.
Solution $\overline{\alpha - \beta} = \overline{(u - x) + i(v - y)} = (u - x) - i(v - y) = (u - iv) - (x - iy) = \overline{\alpha} - \overline{\beta}$.

(c) Prove that $\overline{\alpha \cdot \beta} = \overline{\alpha} \cdot \overline{\beta}$.
Solution $\overline{\alpha \cdot \beta} = \overline{(u + iv)(x + iy)} = \overline{(ux - vy) + i(uy + vx)} = (ux - vy) - i(uy + vx) = (u - iv)(x - iy) = \overline{\alpha} \cdot \overline{\beta}$.

(d) Suppose that $\beta \neq 0$. Prove that $\overline{\alpha/\beta} = \overline{\alpha}/\overline{\beta}$.
Solution We have $(\alpha/\beta)\beta = \alpha$. Apply (iv) so $\overline{(\alpha/\beta)}\overline{\beta} = \overline{\alpha}$. Divide by $\overline{\beta}$ and you have the result.

(e) Suppose that $|\alpha| = 1$. Prove that $\overline{\alpha} = \alpha^{-1}$.
Solution $|\alpha| = \sqrt{x^2 + y^2} = \sqrt{\alpha\overline{\alpha}} = 1$. Squaring we obtain $\alpha\overline{\alpha} = 1$ so $\overline{\alpha} = \alpha^{-1}$.

12. In lectures we discussed the fact that that the following maps from $\mathbb{C}$ to $\mathbb{C}$ are isometries when $\mathbb{C}$ is identified with the Argand diagram: *add a fixed complex number*, *multiply by a complex number of modulus 1* and *complex conjugation*. Composing isometries is, of course, an isometry. We also discussed the fact that any triangle is similar to one in the Argand diagram with vertices at $a, b, c \in \mathbb{C}$ with $|a| = |b| = |c| = 1$, and that in fact we can arrange that $abc = 1$.

(a) Consider the map $\theta : \mathbb{C} \longrightarrow \mathbb{C}$ defined by $\theta(z) = b + c - bc\bar{z}$ for each $z \in \mathbb{C}$. Calculate $\theta(b), \theta(c)$ and $\theta(0)$.
Solution $\theta(b) = b+c-bc\bar{b}$ but $b\bar{b} = 1$ so $\theta(b) = b+c-c = b$. Also $\theta(c) = b+c-b = c$ for similar reasons. Also $\theta(0) = b + c$ which is the reflection of the origin (the circumcentre) in the line through b and c.

(b) Explain why θ is an indirect (orientation reversing) isometry.
Solution θ is obtained as the composition of three maps. First complex conjugate, an orientation reversing isometry. Then multiply by $-(bc)$, a complex number of modulus 1, an orientation preserving isometry, and finally add $b + c$, a translation of the plane which is also an orientation preserving isometry. The composition is therefore an orientation reversing isometry.

(c) Give a geometric interpretation of θ.
Solution We know that θ is an orientation reversing isometry. It fixes b and c and so all points on the line through these points. It does not fix 0, so it is not the identity map. Therefore θ is reflection in the line through b and c.

(d) Let $h = a + b + c$. Show that $t = (a - h)/(c - b)$ is purely imaginary by proving that $\bar{t} = -t$. By cyclically permuting a, b and c in this result, give a geometric interpretation to the point corresponding to the complex number h.
Solution $t = -(b+c)/(c-b) = (b+c)/(b-c)$. Now $\bar{t} = \overline{b+c}/\overline{b-c} = (\bar{b}+\bar{c})/(\bar{b}-\bar{c}) = (ac+ab)/(ac-ab) = (b+c)/(c-b) = -t$. If the triangle is ABC and H is the point corresponding to the complex number $a + b + c$, then $(a - h)/(c - b)$ is purely imaginary means that AH and BC are perpendicular. Similarly, by cyclically permuting letters, we obtain that BH is perpendicular to CA and CH is perpendicular to AB. Therefore H is the orthocentre, the point where the altitudes of triangle ABC meet.

13. Consider a triangle ABC. Its altitudes meet at H, its *orthocentre*. Suppose that a point P lies on the circumcircle of ABC. Consider the three points which are the reflections of P in each of the three sides of ABC. Prove that these points are collinear, and are on a line passing through H. *A solution to the previous question will provide you with the complex number techniques which will allow you to prove this result.*
Solution

The Simson line

Take p (corresponding to the point P) of unit modulus, and reflect it in each of the three sides. The resulting three points and the point H (the orthocentre) corresponding to the complex number $h = a+b+c$ are collinear (a theorem of Wallace) on the "doubled" Simson line. The old-fashioned Simson line, through the vertices of the *pedal triangle* of P (the vertices of which are the feet of the perpendiculars dropped from P to the lines forming the sides of triangle ABC) therefore bisects the line segment PH.

Here is a proof. Use the notation and results of Problem 13. The four points in question are $h, h - (a + \overline{ap})$, and two more similar to the last expression.

Subtracting h, we need to show that $0, a + \overline{ap}$ and two similar expressions are collinear. Write $p = e^{it}$, and let $a = e^{i\alpha}$. Now $a + \overline{ap}$ has argument which is the average of α and $-\alpha - t$, and this is $-t/2$, which is independent of α. Therefore the four points in question are all on the ray with argument $-t/2$ and are thus collinear. Note that ithis shows that if P rotates on the unit circle with uniform speed, then the Simson line rotates at exactly half the speed.

Problem 10 Sheet 8 (Tutor pacifier) Let $ABCDEF$ be a convex hexagon whch has parallel opposite sides and area 1. The lines AB, CD and EF meet in pairs to determine the vertices of a triangle. Similarly, the lines BC, DE and FA meet in pairs to determine the vertices of another triangle. Show that the area of at least one of these two triangles is at least $3/2$. *An classical proof is possible, but you may find some of the ideas developed in this problem* *sheet will assist you to construct a conceptual proof (i.e. a proof consisting of a sequence of ideas, rather than algebraic or trigonometric manipulation).*

The diagram can be viewed as the original hexagon, together with six triangular flaps, opposite flaps being similar. It suffices to show that the total area of the six flaps is at least as great as the area of the hexagon.

Drag A to the origin in the co-ordinate plane by a translation, and then use an invertible linear map to make ABC equilateral. Linear maps preserve ratios of areas, so it suffices to work with the equilateral diagram. Now deploy two moves, each a translation of one of the equilateral triangles in a direction parallel to one of its sides to arrange that the centroids of ABC and DEF co-incide. This has the effect of increasing the area of the hexagon and keeping the area of the flaps constant. It therefore suffices to solve the problem when the centroid co-incide.

All that remains is origami. Suppose, without loss of generality, that ABC has area at least as big as that of DEF. Fold in the three flaps associated with A, B and C. They cover the common centroid because the centroid of an equilateral triangle is 2/3 of the way down an altitude. Now fold in the flaps of DEF, and they fit exactly in the gaps.

Problem 10 Sheet 9 Suppose that $(a_1, a_2), (b_1, b_2), (c_1, c_2) \in \mathbb{Z}^2 \subseteq \mathbb{R}^2$. Giving $\mathbb{R}^2$ the usual geometric interpretation as the Euclidean plane with Cartesian co-ordinates, let the given co-ordinates correspond to the geometric points A, B and C respectively. It so happens that there is no point inside triangle ABC with integral co-ordinates, nor is there any point with integral co-ordinates in the interior of the line segments AB, BC and CA. Determine the area of triangle ABC. *You may assume that ABC is an anticlockwise triangle, in order to avoid getting involved with signed area issues.*

We may translate the triangle so that A is at the origin. The condition concerning the absence of integral grid points is preserved. Both B and C will still have integral grid points. Rotation through π about the midpoint of BC will induce a bijection between $\mathbb{Z}^2$ and $\mathbb{Z}^2$, and sends A to D. Now $ABDC$ is a parallelogram with vertices with integral co-ordinates, and no integral points in its interior or on its sides (other than at the vertices). Let B have co-ordinates (b_1, b_2) and C have co-ordinates (c_1, c_2). The matrix

$$P = \begin{pmatrix} b_1 & c_1 \\ b_2 & c_2 \end{pmatrix}$$

induces a linear map f_P, the determinant of which is the area of parallelogram $ABDC$. This induces a group homomorphism $\widehat{f_P} : \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2$. Now $f_P : \mathbb{R}^2 \longrightarrow R^2$ is bijective, and so injective. Therefore $\widehat{f_P}$ is injective. In fact $\widehat{f_P}$ is also surjective, but that is a more delicate matter. Parallel translations of the parallelogram by the addition to their co-ordinates of expressions of the form $\lambda(b_1, b_2) + \mu(c_1, c_2)$ will result in parallelograms which tile the plane, and all points with integral co-ordinates will arise as vertices (they cannot lie in the interiors, or in the interiors of the sides, of these parallelograms. Thus $\widehat{f_P}$ is surjective and thus bijective. The inverse map to f_P is $f_{P^{-1}}$. Since $\widehat{f_P}$ is bijective, $f_{P^{-1}}$ must map $\mathbb{Z}^2$ to $\mathbb{Z}^2$. In particular $f_{P^{-1}}((1,0)), f_{P^{-1}}((0,1)) \in \mathbb{Z}^2$, so P^{-1} has integral entries. Now $1 = \det P \det P^{-1}$ so $\det P = \det P^{-1} \in \{-1, 1\}$. However, P is orientation preserving, so $\det P > 0$ and so $\det P = 1$. Therefore the area of $ABDC$ is 1 and so the area of ABC is $1/2$.

That argument was quite long, but it was instructive. A faster purely combinatorial proof is available if you know Pick's theorem. See

`http://en.wikipedia.org/wiki/Georg_Alexander_Pick`

and

`http://en.wikipedia.org/wiki/Pick's_theorem.`

It is a discrete version of Green's theorem

`http://en.wikipedia.org/wiki/Green's_theorem`.