1 (a) T
(b) T
(c) F
(d) F
(e) F
(f) T
(g) F
(h) F
(i) T
(j) T

[10]

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2 (a) $f : A \to B$ is injective iff whenever $x, y \in A$ & $f(x) = f(y)$, then $x = y$. [2]

[or if $x, y \in A$ & $x \neq y$, then $f(x) \neq f(y)$.].

(b) $f : A \to B$ is surjective iff whenever $b \in B$, then $\exists\, a \in A$ such that $f(a) = b$. [2]

(c) Choose $x \in A$. Define $g : B \to A$ by:
if $b \in B$ & $\exists\, a \in A$ s.t. $f(a) = b$, then $g(b) = a$
but if $\nexists\, a \in A$ s.t. $f(a) = b$, then $g(b) = x$.
(note that for any $b \in B$, there is at most one $a \in A$ s.t. $f(a) = b$ by injectivity of f.) [2]

(d) $5 \times 4 \times 3 = 60$ [2]

(e) $2^3 - 2 = 6$. [2].

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3 (a) $\mathbb{Z}$ [2]

(b) $\mathbb{Z}_4$ [2]

(c) $\mathbb{Z}_3 \times \mathbb{Z}_3$ with co-ordinatewise operations [2]

(d) $5 = 4 - (-1) = 2^2 - i^2 = (2+i)(2-i)$
& units of $\mathbb{Z}$ are $1, -1, i, -i$. [2]

(e) $\mathbb{Q}$.

4 (a) Possible cycle shapes (**) & (**)(**).

Total number of possibilities is $\binom{5}{2} + 5 \times 3 = 10 + 15$

$= 25.$ [2]

(b) (12) & (345) [2]

(c) (1234) & $(1234)^2$ $(= (13)(24))$ [2]

(1234) ← order 4; $(13)(24)$ ← order 2

(d) $H_1 = \{g \mid g \in G,\ g(4)=4 \text{ & } g(5)=5\}$

$H_2 = \{g \mid g \in G,\ g(1)=1 \text{ & } g(2)=2\}$. [2]

(e) $K_1 = H_1$ from (d); non-abelian

because $(12)(123) = (23)$

but $(123)(12) = (13)$. [2]

$K_2 = \langle (12), (345) \rangle$ which is abelian because (12) & (345) commute.

Both K_1 & K_2 have order 6.

Section B

5. (a) Let $\Omega = \{\lambda m + \mu n \mid \lambda, \mu \in \mathbb{Z}\} \subseteq \mathbb{Z}$

$\Omega \neq \{0\}$ so contains a positive element g.

Let g denote the smallest positive element of Ω.

Let $\omega \in \Omega$, then $\omega = qg + r$ for integers q & r where $0 \leq r < g$.

Now $r = \omega - qg \in \Omega$ so $r = 0$ by minimality of g.

$\therefore g$ divides all elements of Ω & so divides m & n.

If $d \in \mathbb{N}$ & $d \mid m$ & $d \mid n$ then $d \mid \omega$ $\forall \omega \in \Omega$ so $d \mid g$ so $d \leq g$. Therefore g is the g.c.d. of m & n. [4]

(b)
$$29 = 1\cdot 23 + 6$$
$$23 = 3\cdot 6 + 5$$
$$6 = 1\cdot 5 + 1$$

Therefore
$$1 = 6 - 5 = 6 - (23 - 3\cdot 6) = 4\cdot 6 - 23.$$
Now $6 = 29 - 23$ so
$$1 = 4\cdot(29-23) - 23 = (-5)23 + 4\cdot 29$$
(check: $-115 + 116 = 1$ ✓).

$\lambda = -5$, $\mu = 4$ do the job. [3]

(c) $m_1, \dots, m_t$ must be pairwise coprime. [3]

(d) $p_1, \dots, p_{1000}$ are different prime numbers

Let $m_i = p_i^2$, so m_i are pairwise coprime.

Solve $x \equiv -i \bmod m_i$ for $i = 1, \dots, 1000$ simultaneously by CRT.

By CRT $\exists\ n \in \mathbb{Z}$ s.t. $n + i \equiv 0 \bmod m_i$ for $i = 1, \dots, 1000$.

by adding a suitable multiple of $\prod_{i=1}^{1000} m_i$ to n we can make it positive. [3]

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6 (a) $\alpha(xy) = \alpha(x)\,\alpha(y)$ $\forall\ x, y \in G$. [1]

(b) $(xy)^{-1} = x^{-1}y^{-1}$ $\forall\ x, y \in G$

so $y^{-1}x^{-1} = x^{-1}y^{-1}$ $\forall\ x, y \in G$

Invert: $xy = yx$ $\forall\ x, y \in G$ so G is abelian [3]

(c) $o(g) = |\langle g \rangle|$ & $\neq$ so $o(g) \mid |G|$ [3]

by Lagrange's Theorem.

(d) $\exists\ \lambda, \mu \in \mathbb{Z}$ s.t. $\lambda|G| + \mu|H| = 1$. [3]

$\beta(x) = \beta(x^1) = \beta(x^{\lambda|G| + \mu|H|}) = \beta(x^{|G|})^{\lambda} \cdot \beta(x^{\mu})^{|H|}$

$= \beta(1_G)^{\lambda} \cdot \beta(x^{\mu})^{|H|} = 1_H^{\lambda} \cdot \beta(x^{\mu})^{|H|}$

$= 1_H \cdot 1_H = 1_H$.

7 (a) If, for contradiction, S & $P(S)$ in bijective correspondence, then $\exists$

$\theta : S \to P(S)$ a bijection (& so a surjection)

Defn If $x \in S$ & $x \in \theta(x)$, say x is good

If $x \in S$ & $x \notin \theta(x)$, say x is bad.

Every $x \in S$ is good or bad & none is both.

let $Evil = \{ x \mid x \in S, \ x \text{ is bad} \} \subseteq S$

so $Evil \in P(S)$.

Since θ is surjective, there is $y \in S$

s.t. $\theta(y) = Evil$.

If y is good, then $y \in Evil$ so y is bad.

Thus is absurd so y is bad.

Therefore $y \notin Evil$ so y is good. This is absurd //.

[5].

(b) ~~You can set up a bijection between~~.

Suppose that T is uncountable.

Then there are uncountable many sets $\{x\}$ where $x \in T$; these are different & all elements of $F(T)$

so $F(T)$ is uncountable.

Now suppose that T is countable. If T is finite then $F(T)$ is finite (& so countable).

Otherwise T is in bijective correspondence with $\mathbb{N}$.

We proved in the course that $F(\mathbb{N})$ is countable.

Therefore T is uncountable iff $F(T)$ is uncountable.

[5].