Algebra 1A 2012 Solutions (Examined January 2013)

1 (a) F (consider $\emptyset$)

(b) F

(c) F

(d) T

(e) F ($\{\{r\} \mid r \in \mathbb{R}\}$ is uncountable)

(f) F ($\{r \mid r \in \mathbb{R}, r \geq 0\}$ is a transversal)

(g) T

(h) T

(i) F

(j) F ($91 = 13 \times 7$)

$\boxed{\text{1 mark each}}$

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2 (a) $f \circ g$ a bijection $\Rightarrow$ f surjective & g injective

$g \circ f$ a bijection $\Rightarrow$ f injective & g surjective

$\therefore$ f & g are both bijective. $\boxed{2}$

(b) No, it does not follow.

For example $f_2 : \mathbb{Z} \to \mathbb{Z}$ is defined by $x \mapsto 2x \ \forall x \in \mathbb{Z}$

$f_1 : \mathbb{Z} \to \mathbb{Z}$ is defined by $x \mapsto \lfloor \frac{x}{2} \rfloor \ \forall x \in \mathbb{Z}$

(this is the 'floor' or 'integer-part' function $\lfloor \ \rfloor$ defined on $\mathbb{R}$ by $\lfloor y \rfloor = \max\{x \mid x \in \mathbb{Z}, x \leq y\}$)

Now $f_1 \circ f_2 = \mathrm{Id}_{\mathbb{Z}}$

but $(f_2 \circ f_1)(0) = 0$ & $(f_2 \circ f_1)(1) = 0$

so $f_2 \circ f_1$ is not a bijection (it is not an injection).

$\boxed{5}$

(c) There are 8 maps from $\{1,2,3\}$ to $\{1,2\}$, all of which are surjective except the two constant maps α_1 & α_2. Here $\alpha_1(x) = 1 \ \forall x \in \{1,2,3\}$ & $\alpha_2(x) = 2 \ \forall x \in \{1,2,3\}$

Therefore there are $8-2 = 6$ surjections. $\boxed{2}$

(d) There are $6 \times 5 \times 4 \times 3 = 360$ such injections.

This is because there are 6 choices as to where to send 1, then 5 choices as to where to send 2 etc. $\boxed{3}$

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3 $A = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}$

(a) $A^2 = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ 4 & -1 \end{pmatrix}$ $\boxed{2}$

(b) $A^T = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}$ $\boxed{2}$

(c) $A^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{pmatrix}$ $\boxed{2}$

(d) $A + A^T = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ $\boxed{2}$

(e) $(A + A^T)^T = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ $\boxed{2}$

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4 (a) The elements of order 2 must be either transpositions or products of disjoint transpositions. $\boxed{3}$

There are $\binom{4}{2} + 3 = 6 + 3 = 9$ elements of order 2

(b) These elements commute with (12):

$\{ id, (12), (34), (12)(34) \}$ & no others

either by inspection or by letting G act on itself by conjugation — then the orbit of (12) will have size $6 = \binom{4}{2}$ So the centralizer = stabilizer of (12) will have size 4 (i.e. our list of elements). The answer is ~~therefore~~ 4.

$\boxed{3}$

(c) The elements of G which commute with $(12)(34)$ are

$\{ id, (1324), (1324)^2 = (12)(34), (1324)^3 = (1423),$

$(12), (34), (13)(24), (14)(23) \}$.

You expect that there should be 8 such elements because of the orbit - stabilizer theorem: G acts on itself by conjugation & the conjugates of $(12)(34)$ are $\{ (12)(34), (13)(24), (14)(23) \}$. $\boxed{4}$

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5 (a)

$$6647 = 4 \times 1411 + 1003$$
$$1411 = 1 \times 1003 + 408$$
$$1003 = 2 \times 408 + 187$$
$$408 = 2 \times 187 + 34$$
$$187 = 5 \times 34 + 17$$
$$34 = 2 \times \underline{\underline{17}} + 0$$

$\therefore \gcd(1411, 6647) = 17$ $\boxed{2}$

(b)

$$17 = 1 \cdot 187 - 5 \cdot 34$$
$$= 11 \cdot 187 - 5 \cdot 408$$
$$= 11 \cdot 1003 - 27 \cdot 408$$
$$= 38 \cdot 1003 - 27 \cdot 1411$$
$$= 38 \cdot 6647 - 179 \cdot 1411$$

$\boxed{3}$

(c) The positive integers $n_1, n_2, \ldots, n_6$ must be pairwise coprime. $\boxed{2}$

(d) We need to solve the simultaneous congruences:

$$n \equiv 9 \mod 11$$
$$n \equiv 11 \mod 13$$
$$n \equiv 13 \mod 15$$
$$n \equiv 15 \mod 17$$

$\boxed{3}$

Note the moduli are pairwise coprime so Chinese Remainder Theorem applies. Notice -2 is a simultaneous solution. So set of solutions is $\{ -2 + \lambda\, 11 \cdot 13 \cdot 15 \cdot 17 \quad \lambda \in \mathbb{Z} \}$

So smallest positive solution is $11 \cdot 13 \cdot 15 \cdot 17 - 2$.

6 (a) If $f, g \in \mathbb{R}[X]$ & $g \neq 0$,
then $\exists\, q, r \in \mathbb{R}[X]$ such that $f = qg + r$
with $\deg r < \deg g$ (N.B. $\deg 0 = -\infty$)
This means you can run a Euclid's algorithm
(just as for $\mathbb{Z}$) to calculate a gcd of f & g $\boxed{3}$
(& can make it monic by multiplying by the
inverse of its leading coeff.).
The reasons this works are the same as for $\mathbb{Z}$. $\boxed{2}$

(b) Unwrapping the E.A. calculation, you
can calculate $\lambda, \mu \in \mathbb{R}[x]$ s.t. $\lambda f + \mu g = h = \gcd(f, g)$.

(c) h, λ, μ, f, g as above.
If α is a common root of f & g then

$$h(\alpha) = \lambda(\alpha) f(\alpha) + \mu(\alpha) g(\alpha) = \lambda(\alpha).0 + \mu(\alpha).0 = 0$$

so α is a root of h.
On the other hand, if β is a root of h,
& $h = \gcd(f, g)$, then $\exists\, l, m \in \mathbb{R}[x]$
s.t. $f = l.h$ & $g = m.g$
so $f(\beta) = l(\beta)h(\beta) = 0 = m(\beta)h(\beta) = g(\beta)$.
There fore γ is a root of h (in $\mathbb{C}$)
if, and only if, γ is both a root of f
& a root of g. $\boxed{5}$

7(a) G a finite group & $H \leq G$, then $|H| \,\big|\, |G|$. $\boxed{2}$

(b) $G = S_5 = \mathrm{Sym}(\{1,2,3,4,5\})$.

$a = (12)$, $b = (34)$, $c = (15)$. $\boxed{3}$

Clearly $(12)(34) = (34)(12)$ & $(34)(15) = (15)(34)$.

However $(12)(15) = (152)$

but $(15)(12) = (125)$. $\boxed{2}$

(c) α is a homomorphism iff $\forall x, y \in G$, $\alpha(xy) = \alpha(x)\alpha(y)$.

(d) You need to assume that $|G|$ is finite.

By Lagrange's theorem it suffices to prove that $K \leq G$.

We do this: $1 \in K$ since $\alpha(1_G) = 1_H = \beta(1_G)$ so $K \neq \emptyset$.

Now suppose that $k, l \in K$, so $\alpha(l) = \beta(l)$ & $\alpha(k) = \beta(k)$

$$
\begin{aligned}
\text{Now } \alpha(l^{-1}k) &= \alpha(l^{-1})\,\alpha(k) && (\alpha \text{ is a hom}^m) \\
&= \alpha(l)^{-1}\,\alpha(k) && (\text{property of hom}^m) \\
&= \beta(l)^{-1}\,\beta(k) && (k, l \in K) \\
&= \beta(l^{-1})\,\beta(k) && (\text{property of hom}^m) \\
&= \beta(l^{-1}k) && (\beta \text{ is a hom}^m)
\end{aligned}
$$
$\boxed{3}$

$\therefore l^{-1}k \in K$. Therefore $K \leq G$.

8 $\underline{p} = (p_1, p_2)$, $\underline{q} = (q_1, q_2) \in \mathbb{R}^2$ & $\underline{p} \neq \underline{q}$

The line through these points is

$\{ \underline{p} + t(\underline{q} - \underline{p}) \mid t \in \mathbb{R} \}$

(a) $\{ \alpha(\underline{u}) \mid \underline{u} \in L \mid t \in \mathbb{R} \}$ $\boxed{3}$

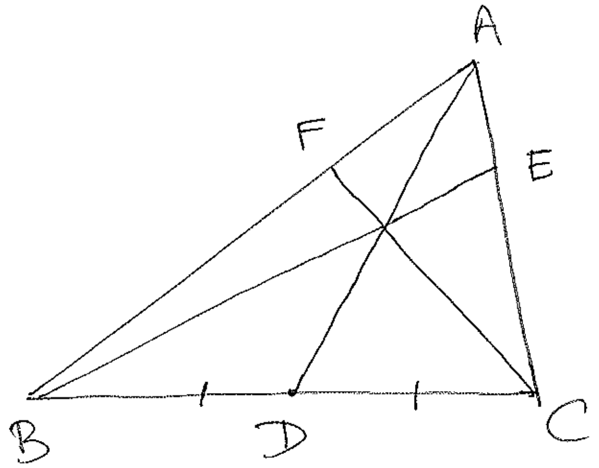
$= \{ \alpha(\underline{p}) + t(\alpha(\underline{q}) - \alpha(\underline{p})) \mid t \in \mathbb{R} \}$

is the line through $\alpha(\underline{p})$ & $\alpha(\underline{q})$.

(b) The midpoint of PQ is $\frac{1}{2}\underline{p} + \frac{1}{2}\underline{q}$. $\boxed{3}$

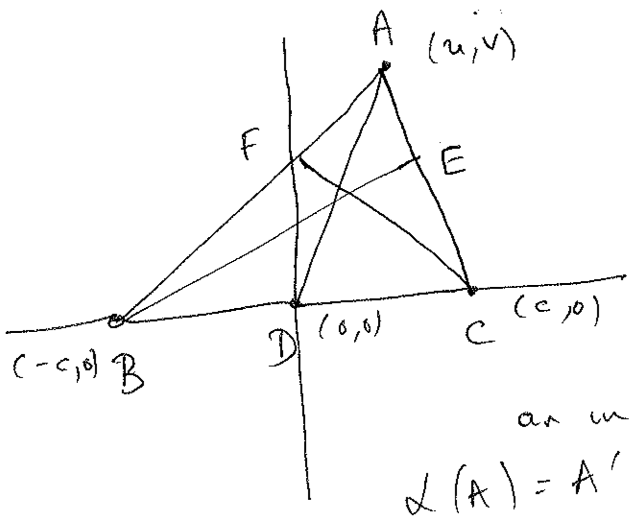
This maps the $\frac{1}{2}\alpha(\underline{p}) + \frac{1}{2}\alpha(\underline{q})$ which is the midpoint of $\alpha(P)\alpha(Q)$.

(c) $\boxed{4}$



Choose origin at D, "x-axis" DC

define



$\alpha : \mathbb{R}^2 \to \mathbb{R}^2$

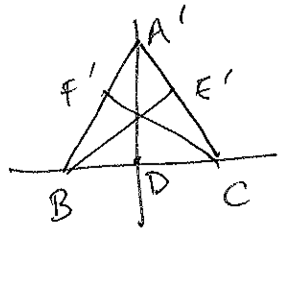
via matrix $\begin{pmatrix} 1 & -v^{-1}u \\ 0 & 1 \end{pmatrix}$

& $(x, y) \mapsto (x - v^{-1}u\,y, y)$

an invertible linear map.

$\alpha(A) = A' = (0, v)$, $\alpha(B) = B' = B$, $\alpha(C) = C' = C$.

Image.



by symmetry of isosceles Δ, $F'E' \parallel BC$

i.e. $F'E' \cap BC = \emptyset$. Now apply α^{-1} so $FE \cap BC = \emptyset$ $\therefore FE \parallel BC$. //