Symmetric groups and cycle notation

GCS

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Symmetric Group

Suppose that Ω is a set. Let $\mathrm{Sym}(\Omega)$ denote the set

$$\{f \mid f : \Omega \to \Omega, f \text{ is bijective } \}$$

made into a group by using composition of maps as the group operation. This group is called *the symmetric group on* Ω.

For example, perhaps $\Omega = \mathbb{Z}$, in which case $\mathrm{Sym}(\mathbb{Z})$ has some interesting subsets. Please verify that they are all subgroups:

(i) The rigid motions subgroup:

$$\{f \mid f \in \mathrm{Sym}(\mathbb{Z}), |f(i) - f(j)| = |i - j| \ \forall i, j \in \mathbb{Z}\}.$$

(ii) The very idle subgroup (in which almost all integers like to stay in bed):

$$\{f \mid f \in \mathrm{Sym}(\mathbb{Z}), f(i) = i \text{ for all except for finitely many integers } i\}.$$

(iii) The lazy subgroup (where numbers do not like exercise):

$$\{f \mid f \in \mathrm{Sym}(\mathbb{Z}), |f(i) - i| \text{ is bounded } \}.$$

(iv) Let n be a natural number. The mod n fan club subgroup is

$$\{f \mid f \in \mathrm{Sym}(\mathbb{Z}), \text{ if } i \equiv j \bmod n, \text{ then } f(i) - i = f(j) - j\}.$$

(v) The *Glastonbury* group is the union of the mod n fan club subgroups (over all possible positive integers n)

Describe which of these groups is a subgroup of another of these groups.

Cycle Notation

When $\Omega = \{1, 2, \dots, n\}$, the group $\mathrm{Sym}(\Omega)$ is often written S_n and is called the symmetric group of degree n. Notice that $|S_n| = n!$. We now carefully explain cycle notation. To fix ideas, let $n = 5$ so $S_5 = 120$. If $f \in S_5$ we list $1, f(1), f(f(1)), \dots$ until we reach an element already mentioned. Now f is injective so this first repeated element must be 1. We put this list in round brackets. Thus if $f(1) = 3, f(3) = 5$ and $f(5) = 1$ we write $(1, 3, 5)$ [omit the commas if you like]. The question arises, what about 2? If $f(2) = 4$, then it follows that $f(4) = 2$ and the full description of f in cycle notation is $(1, 3, 5)(2, 4)$. On the other hand, if f fixes 2, then it must fixed 4 too, and one might write $f = (1, 3, 5)(2)(4)$. However, we usually just omit the fixed points, and write $f = (1, 3, 5)$.

The 120 elements of S_5 have various cycle shapes. There is the identity. Then there are $\binom{5}{2} = 10$ transpositions such as $(1, 2)$. There are $\frac{1}{2}\binom{5}{2}\binom{3}{2} = 15$ elements which are the product of disjoint 2-cycles such as $(1, 2)(3, 4)$. There are $2\binom{5}{3} = 20$ 3-cycles such as $(1, 2, 3)$. There are also $2\binom{5}{3} = 20$ products of 3-cycle and a disjoint transposition such as $(1, 2, 3)(4, 5)$. There are $5(3!) = 30$ 4-cycles such as $(1, 2, 3, 4)$ and there are $4! = 24$ 5-cycles such as $(1, 2, 3, 4, 5)$. Notice that

$$1 + 10 + 15 + 20 + 20 + 30 + 24 = 120 = 5!$$

and we have found all the elements of S_5.

Inverses

Notice that the inverse of an element of S_n is obtained by reversing each cycle. The inverse of the element $(1,2)(3,4,5)(7,9)$ is $(2,1)(5,4,3)(9,7)$ which you can also write as $(1,2)(3,5,4)(7,9)$. Notice that an element of S_n which is its own inverse is either the identity element or a product of disjoint 2-cycles such as $(1,2)(3,4)(7,9)$.

Products

This is how to multiply elements in cycle notation: consider

$$(1,2)(3,4,5)(7,8) \cdot (1,2,3)(6,7,8)$$

a product in S_{10}. First see what happens to 1; its destiny is $1 \mapsto 2 \mapsto 1$ so 1 is a fixed point of the composed map. Now $2 \mapsto 3 \mapsto 4$ so we write $(2,4$. Now we see what happens to 4. We have $4 \mapsto 4 \mapsto 5$ so we write $(2,4,5$. Next see what happens to 5. We have $5 \mapsto 5 \mapsto 3$ so we write $(2,4,5,3$. Next see what happens to 3. We have $3 \mapsto 1 \mapsto 2$ so we write $(2,4,5,3)$. Next see what happens to 6. We have $6 \mapsto 7 \mapsto 8$ so we write $(2,4,5,3)(6,8$. Next see what happens to 8. We have $8 \mapsto 6 \mapsto 6$ so we write $(2,4,5,3)(6,8)$. Notice that 7, 9 and 10 are all fixed points of the composed maps so the final description of this composed bijection is $(2,4,5,3)(6,8)$

Support

The *support* of an element f of a symmetric group is the subset of Ω consisting of all $w \in \Omega$ such that $f(w) \neq w$. Thus the support of the identity map is $\emptyset$. Notice that if $f, g \in \mathrm{Sym}(\Omega)$ have disjoint supports (i.e. the intersection of their supports is empty) then f and g commute. In particular

$$(12) \cdot (34) = (34) \cdot (12) = (12)(34).$$

However, whilst this condition is suffient for elements to commute, it is not necessary. For example, in S_n where $n \geq 4$, the elements $(1,2)(3,4)$, $(1,3)(2,4)$ and $(1,4)(2,3)$ all commute. Indeed, the product of any two of them is the third. Together with the identity, they form a subgroup of S_4 which you might call the rectangle group. Take a rectangle which happens not to be a square, and label the corners 1,2,3 and 4 in cyclis order. The rigid symmetries of the rectangle consist of the identity, rotating it through π about an axis through and perpendicular to its centre, and two more elements which are each a reflection in one of its two axes of symmetry (turn it over).

Orders of elements

When G is a group and $g \in G$, it may happen that some positive integer m has the property that $g^m = id$. If so, the smallest such m is called the *order* of g and written $o(g)$. If no such m exists we say that g has infinite order.

Suppose that $f \in S_n$ for some n. If f is a $k-cycle$, then $o(f) = k$. Recall that elements with disjoint support commute. Therefore if f is written in cycle notation, and has cycles of length $k_1, k_2, \ldots, k_t$ (fixed points are cycles of length 1). Then $o(f)$ is the least common multiple of $k_1, k_2, \ldots, k_t$. In particular, all elements of S_n have finite order.