Celebrity bijections

GCS

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Bijections

If you have a bijection $f\colon A \to B$, then there is a unique map $g\colon B \to A$ such that $g \circ f = Id_A$ and $f \circ g = Id_B$. The map g is also a bijection.

Making Bijections

It is always attractive to have an inverse for a map, but if the map is not bijective, then you do not have one. You can "improve" any map and make it a surjection by discarding elements of the codomain which are not in the image of the function. You can then discard elements from the domain to make it a bijection. This is often done to celebrity maps which are not bijections, so that you have some sort of inverse.

Squaring

Consider the map $s : \mathbb{R} \to \mathbb{R}$ defined by $x \mapsto x^2\ \forall x \in \mathbb{R}$. This map is neither injective nor surjective. You can make it surjective by discarding the negative real numbers from the codomain. We can also impose injectivity by discarding the negative real numbers from the domain. Let us use interval notation beloved of the analysts:

Consider the map $\widehat{s}\colon [0,\infty) \to [0,\infty)$ defined by $\widehat{s}(x) = s(x)(= x^2)\forall x \in [0,\infty)$. This map $\widehat{s}$ is a bijection, and it has a unique inverse function usually denoted $\sqrt{\ }$. Thus

$$\sqrt{\ }\colon [0,\infty) \to [0,\infty)$$

is defined by, for each $x \in \mathbb{R}$, $\sqrt{x}$ is the unique element of $[0,\infty)$ such that $(\sqrt{x})^2 = x$.

Tangent

The (pathetic excuse for a) map $\tan : \mathbb{R} \to \mathbb{R}$ is actually not a map, because it is not defined at odd multiples of $\pi/2$. However you can easily make it into a map by restricting the domain, and regard it as a map $\tan\colon (-\pi/2, \pi/2) \to \mathbb{R}$. This is now a (respectable) map, and is a bijection.

This is interesting because it shows that the sets $(-\pi/2, \pi/2)$ and $\mathbb{R}$ are in bijective correspondence. The inverse map $\tan^{-1}\colon \mathbb{R} \to (-\pi/2, \pi/2)$ is important in the calculus because

$$\int \frac{1}{1+x^2} dx = \tan^{-1} x + C.$$

Sine and Cosine

These maps are not bijective if you regard them as maps from $\mathbb{R}$ to $\mathbb{R}$. They can be made into bijections by being careful with the domains and codomains. Thus one can consider $\sin\colon [-\pi/2, \pi/2] \to [-1, 1]$ and $\cos\colon [0, \pi] \to [-1, 1]$ which are both bijections. The inverse maps of these bijections are also important in the calculus.