The Anatomy of A_5

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Symmetric and Alternating Groups

Recall that the *symmetric group* on a set Ω is the set of bijections from Ω to Ω with composition of maps as the group law. This group is denoted in various ways in the literature, most commonly as $\mathrm{Sym}(\Omega)$ and $\mathrm{Sym}\,\Omega$. It is clear that any additional structure that Ω might have (it could be a group or set of cabbages) is irrelevant, and if Ω' is a set in bijective correspondence with Ω, then $\mathrm{Sym}(\Omega)$ and $\mathrm{Sym}(\Omega')$ will be isomorphic groups.

In the event that Ω is finite and is a set of size (cardinality) n, then we may as well assume that $\Omega = \{1, 2, \ldots, n\}$. In this case we often write $\mathrm{Sym}(\Omega)$ is $\mathrm{Sym}(n)$ or S_n. Thus $|S_n| = n!$. Notice that S_n has n subgroups, each of which is a copy of S_{n-1}, obtained by considering those elements of S_n which have i as a fixed point (for $1 \le i \le n$).

We have introduced cycle notation for elements of S_n, and that is discussed in lecture notes.

We make the assumption from now on that the integer n is at least 2.

Recall that an *even* permutation is an element of S_n which is the product of an even number of transpositions (ij), and an *odd* permutation is an element of S_n which is the product of an odd number of transpositions (ij). It is not immediately obvious, but we proved it in lectures, that a permutation cannot be both even and odd. Left multiplication by a chosen transposition induces mutually inverse bijections between the even and odd permutations of S_n. Thus there are $n!/2$ even permulations, and it is a routine matter to verify that they form a group $\mathrm{Alt}(n)$ or A_n. Therefore A_n and $(ij)A_n$ are the (disjoint) left cosets of A_n in S_n whenever (ij) is a transposition, and $(ij)A_n$ is the set of odd permutations. For similar reasons, $A_n(ij)$ is also the set of odd permutations in S_n.

Exercise

1. If π is an even permutation, then $\pi A_n = A_n = A_n\pi$.

2. If π is an odd permutation, then $\pi A_n = A_n\pi$ This is the set of odd permutations.

3. Deduce that $gA_n = A_n g$ for every $g \in S_n$.

The anatomy of A_5

Notice that the order of A_5 is $5!/2 = 60$. We classify its elements. It has the identity element id, 15 elements which are products of disjoint transpositions such as $(12)(34)$. There are 15 of them because you can choose the unmentioned fifth number (in our case 5) in 5 ways, and then use the remaining four numbers to make a product of two disjoint (non-overlapping) transpositions in 3 ways, and $3 \times 5 = 15$. Notice that $(12)(34) = (34)(12)$, and we do not have to worry about the order in which the two disjoint transpositions are written.

So far we have found $1 + 15$ elements of A_5. Notice that $(13)(12) = (123)$ and more generally if i, j, k are different, then $(ik)(ij) = (ijk)$. Thus all 3-cycles (ijk) are in A_5. Notice that $(ijk) = (jki) = (kij)$. We can choose the two fixed points of 3-cycle in $\binom{5}{2} = 10$ ways, and use the remaining 3 numbers to make exactly 2 different (mutually inverse) 3-cycles. Thus A_5 contains $2 \times 10 = 20$ 3-cycles and we have found $1+15+20 = 36$ elements of A_5.

Notice that $(15)(14)(13)(12) = (12345)$ is an even permutation, and for similar reasons all 5-cycles are even permutations. Any 5-cycle can be written in a unique way if we insist that 1 is on the left, so there are $4! = 24$ 5-cycles, so we have found all $60 = 1+15+20+24$ elements of A_5. Notice that the order of the product of disjoint transpositions is 2, the order of a 3-cycle is 3 and the order of a 5-cycle is 5.

The possible orders of subgroups of A_5 are $1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30$ and 60 by Lagrange's theorem, and some of these possibilities arise. The trivial subgroup $\langle \mathrm{id} \rangle = \{\mathrm{id}\}$ has order 1, and $\langle (12)(34) \rangle$ is a cyclic group of order 2. Then $\langle (123) \rangle$ is a cyclic group of order 3 and $\langle (12345) \rangle$ is a cyclic group of order 5. The elements of A_5 which fix 5 form a subgroup which is a copy of A_4 and so will have order 12. Also A_5 itself is a subgroup of order 60. There are less obvious subgroups such as this copy of the Klein 4-group

$$\{\mathrm{id}, (12)(34), (13)(24), (14)(23)\}$$

of order 4.

We now construct a subgroup of order 6. Take each element of S_3 (of which there are 6) and, if it is an odd permutation, multiply it by (45) to make it into an even permutation. We get the following 6 permutations

$$\mathrm{id}, (12)(45), (13)(45), (23)(45), (123), (132).$$

which form a group isomorphic to S_3. All the interesting behaviour happens as $1, 2$ and 3 are moved around, with the extra padding transposition (45) twinkling in and out of existence so as to keep the permutations even. Therefore A_5 contains a subgroup of order 6.

We have discussed dihedral groups. If you label the vertices of a regular pentagon (5-gon) in cyclic anticlockwise order 1,2,3,4,5, then anticlockwise rotation through $\frac{2\pi}{5}$ yields the permutation (12345) of the vertices. Reflection in the axis of symmetry through the vertex 1 yields the permutation (23)(45). We obtain a subgroup of A_5 which is a copy of the dihedral group D_{10}: consists of all 5 powers of (123454) together with 5 elements of order 2 which are not transpositions. These are the reflections $(25)(34), (13)(45), (15)(24), (12)(35)$ and $(14)(23)$. Thus A_5 has a subgroup of order 10.

That leaves 15, 20 and 30 as candidates for sizes of subgroups of A_5. In fact there are no subgroups of these orders, as we now demonstrate. Suppose (for contradiction) that H were such a subgroup of order 15 or 20. There are 24 elements of order 5 in A_5, so at least one of them, say x, is not in H. The cosets H, xH, x^2H, x^3H, x^4H cannot all be different since H has index 4 or 3 in A_5. Therefore $x^iH = x^jH$ for some $0 \leq i < j \leq 4$ so $H = x^{j-i}H$ and so $x^{j-i} \in H$ for those i and j. Now $\langle x^{j-i} \rangle$ is a subgroup of $\langle x \rangle$ and, since it is not the trivial subgroup, Lagrange's theorem applies and so $\langle x^{j-i} \rangle = \langle x \rangle$ (5 is prime). Therefore $x \in H$ which is absurd since $x \notin H$.

Finally we suppose, for contradiction, that A_5 has a subgroup of order 30. Since A_5 has $24+20$ elements of order 3 and 5, there must be at least one, say y which is not in H. Now replicate the previous argument by looking at the cosets y^iH for $i = 0, 1, 2$ (since H only has two left cosets in A_5, two of these three left cosets must be the same), and obtain the absurd concusion that $y \in H$.

Therefore A_5 has three positive divisors (15, 20 and 30) which do not arise as orders of subgroups.

The Alternating Group A_5 at Christmas

A regular dodecahedron is a solid figure which has 12 faces, each of which is an identical regular pentagon. It has 20 vertices, and makes an excellent Christmas Tree bauble. It has a group of rigid symmetries of order 60 (not including reflections). To see this, place such a bauble on the ground. Pick it up, move it around and replace it to occupy the same space as when you picked it up. It can place it on any one of its 12 faces as a base in any one of 5 positions, so this group has order 60. Perhaps it should not be a surprise that this group is an isomorphic copy of A_5. You might think that if this is so, then there should be 5 “things” which the group is permuting. This is so. Please use an internet search engine on: regular dodecahedron 5 cubes (and look for images).

Exercises

1. Play with A_4 and understand its subgroups.

2. Suppose that H is a subgroup of A_5 such that $xH = Hx$ for every $x \in A_5$. Prove that H is either the trivial subgroup (of order 1) or A_5.