Mathematical Entertainments (Solutions)

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1. *There are 100 diamonds in a display case. Fifty are genuine and fifty are fake. Ellen the expert can tell which are fake and which are genuine. You wish to purchase the 50 genuine diamonds. The consultation rules are that you can point to any three diamonds, then Ellen will temporarily cover one of these three diamonds, and truthfully tell you how many of the other two diamonds are genuine. You can repeat this procedure as many times as you wish. Is it possible for you to determine which 50 diamonds are genuine, no matter how Ellen decides to answer your queries (remember she can select which diamond to cover at every stage).*
   Solution No it is not possible. Ellen can prevent you from determining the genuine and fake diamonds as follows. She selects a particular genuine diamond and a particular fake diamond. Call them G and F respectively. Whenever she is asked about three stones which include exactly one of F and G, she hides that diamond. When she is asked about three diamonds which include both F and G, she hides the third diamond and answers "one". When she is asked about three diamonds which include neither F nor G, she covers a diamond at random and answers correctly concerning the other two. You cannot distinguish F from G because the answers Ellen gives would be exactly the same had F been genuine and G been fake.

2. *Find all prime numbers a and b such that $20a^3 - b^3 = 1$.*
   Solution 1 Suppose that a and b are prime numbers which render the equation correct, then

$$20a^3 = (b+1)(b^2 - b + 1) = (b+1)(b(b+1) - 2(b+1) + 3).$$

   Any positive common factor of the two factors on the right must divide 3. However, if 3 were a common factor, then 3 would divide a, so the prime number a would be 3. However 539 is not a cube so this is impossible. It follows that $b+1$ and $b^2 - b + 1$ have no prime factor in common. Now $b^2 - b + 1$ is odd so 4 divides $b+1$. Also $b^2 - b + 1$ cannot be a multiple of 5 (look at $b = 5k, 5k+1, 5k+2, 5k+3$ and $5k+4$ in turn). Therefore $b+1$ is a multiple of 5 and so of 20. Now if a (and therefore a^3) were to divide $b+1$, then this would force $b^2 - b + 1 = 1$, but $b^2 - b + 1 = (b-1)^2 + b \geq b > 1$

because $b > 1$. Thefore $b^2 - b + 1 = a^3$ and $b + 1 = 20$. Therefore $b = 19$ and so $a = 7$. Now 7 and 19 are prime numbers, and do satisfy $20a^3 - b^3 = 1$ so we have found a solution, and moreover we have proved that it is the only solution (i.e. it is the unique solution).

Solution 2 Any cube of an integer must leave remainder $-1, 0$ or 1 on division by 7. You can verify this by cubing $7k + i$ for $i = 0, 1, \ldots, 6$ with k an integer. In other words, **every cube is of the form** $7k - 1, 7k$ **or** $7k + 1$ **for a suitable integer** k. Now $20a^3 - b^3 = 21a^3 - (a^3 + b^3) = 1$ leaves remainder 1 on division by 7, so $a^3 + b^3$ leaves remainder -1 on division by 7. Now by the remark in bold font, one of a^3 or b^3 is a multiple of 7, so $a = 7$ or $b = 7$ are the only possibilities since a and b are prime numbers. Now $7^3 + 1 = 344$ is not a multiple of 20 so $b \neq 7$. If $a = 7$, then $b^3 = 20a^3 - 1 = 6859 = 19^3$. Now 7 and 19 are prime, and do satisfy $20a^3 - b^3 = 1$, and we have proved that these are the only prime number solutions of this equation.

3. *Each integer is coloured red or blue according to the following rules.*

   (a) *The number 1 is red.*

   (b) *If a and b are two red numbers, not necessarily different, then $a - b$ and $a + b$ are of different colour.*

   Prove that there is exactly one way to colour the integers according to these rules, and determine the colour of 2017.

   Solution All numbers which are multiples of 3 must be coloured blue, and all other numbers must be coloured red. First we check that this is a valid colouring. Certainly 1 is red so (a) is satisfied. If two numbers are both coloured red, then each of them either (i) leaves 1 on division by 3, (ii) leaves remainder 2 on division by 3 or (iii) one leaves remainder 1 and the other leaves remainder 2 on division by 3. In cases (i) and (ii), their difference is blue and their sum is red. In case (iii), then their sum is blue but their difference is red. Therefore the colouring we have suggested does indeed satisfy the two given conditions.

   Next we will establish that our colouring is the unique solution. First we know that 1 is red. Therefore 0 and 2 are of different colour. However, 0 cannot be red, for then $0 + 0$ and $0 - 0$ would be of different colour which is absurd. Therefore 0 must be blue and 2 must be red. We now show (by induction) that all other integers have their colours forced. Here is the inductive argument. Suppose that, for some integer k, we have $3k$ is blue, $3k + 1$ is red and $3k + 2$ is red.

   Now 2 is red, $3k + 2$ is red but $3k$ is blue, so $3k + 4 = 3(k + 1) + 1$ is red. Next 1 is red, $3k + 2$ is red but $3k + 1$ is red so $3k + 3 = 3(k + 1)$ is blue. Next 1 is red, $3k + 4$ is red but $3k + 3$ is blue so $3k + 5 = 3(k + 1) + 2$ is red. This paragraph enables us to induct "upwards" through the positive integers, so we obtain the predicted colouring on the positive integers.

Next we must develop the machinery to induct "downwards", Recall that we are assuming that $3k$ is blue, $3k + 1$ is red and $3k + 2$ is red. Now $3k+1$ is red and 2 is red but $3k+3$ is blue so $3k-1$ is red. Next $3k-1$ is red and 1 is red, but $3k$ is blue so $3k-2$ is red. Finally $3k-2$ is red and 1 is red, but $3k-1$ is red so $3k-3 = 3(k-1)$ is blue. We can now induct both up and down, so the red and blue pattern specified at the outset is the only solution.

Do not publish these problems on the internet, or indeed anywhere else. Solutions will be available at the end of week 1.