MA10209 Algebra 1A

Counting Partitions: GCS

30-x-11

The course website is http://people.bath.ac.uk/masgcs/diary.html

This was Problem 3 of Sheet 3.

3 Let $I_n = \{1, 2, 3, \ldots, n\}$. For $n = 1, 2, 3, 4$ and 5, determine the number of partitions of $I_n = \{1, 2, 3, \ldots, n\}$.
Solution

$n = 1$ There is a unique partition $\{1\}$.

$n = 2$ There are two partitions: $\{1, 2\}$; $\{1\}$, $\{2\}$

$n = 3$ There are 5 partitions: one of shape 1,1,1; 3 of shape 2,1 and 1 of shape 3.

$n = 4$ There are 15 partitions: one of shape 1,1,1,1; 6 of shape 2,1,1; 3 of shape 2,2; 4 of shape 3,1 and 1 of shape 4.

$n = 5$ There are 52 partitions: one of shape 1,1,1,1,1; 10 of shape 2,1,1,1; 15 of shape 2,2,1; 10 of shape 3,1,1; 10 of shape 3,2; 5 of shape 4,1 and one of shape 5.

We elaborate on enumerating partitions. Suppose that you have five people, and you wish to put them into three teams: a blue team of two people, a red team of two people and white team of one person. There are 30 ways to do this. You can select the members of the blue team in $\binom{5}{2} = 10$ ways. Having done that, there are three people left. You can select the person for the white team in $\binom{3}{1} = 3$ ways. Once you have done that, the members of the red team are forced. Altogether there are $10 \times 3 = 30$ ways to select the teams.

Now ask a slightly different question. How many ways are there to partition five people into three teams, one of size 2, another of size 2, and a third of size 1? The answer 30 is now incorrect, because we don't care about which team is which. The method of the previous paragraph overcounts the partitions by a factor of 2 (actually 2!), and the correct answer is 15.

Here is a general method. Suppose that the teams are named (or coloured), and you must select teams of size $r_1, r_2, \ldots, r_t$ from a collection of $n = r_1 + r_2 + \cdots r_t$ people. This is

$$\binom{n}{r_1} \cdot \binom{n - r_1}{r_2} \cdot \binom{n - r_1 - r_2}{r_3} \cdots \binom{r_t}{r_t}$$

but if you write out the appropriate formulas for binomial coefficients and cancel, you get a tidy answer

$$\frac{n!}{r_1! r_2! \cdots r_t!}.$$

In the case of the five people being allotted to coloured teams, this gives $5!/(2!2!1!) = 120/4 = 30$. as required.

If the teams are not named (or coloured), but you know that of the t teams, s_1 must have one size, s_2 must have another size etc until s_k must have a final size (so $s_1 + s_2 + \cdots + s_k = t$), then the number of ways of doing this is

$$\frac{n!}{(r_1!r_2!\cdots r_t!)(s_1!s_2!\cdots s_k!)}.$$

In our concrete example involving five people, we must divide 30 by 2!1!.

Let us look at a more daunting example. Suppose that you want to count partitions of 10 into four subsets of size $3, 3, 2$ and 2. Using the theory we have developed this is

$$\frac{10!}{(3!3!2!2!)(2!2!)} = 6300.$$