Club Sheet 3

solutions to supplementary problems: GCS

18-xi-16

1. Let n be a positive integer. Show that the sum of the largest odd divisors of $n+1$, $n+2$, $\ldots$, $2n$ is a perfect square.
   Solution We introduce the term the *odd part* of a non-zero integer m to denote the largest positive integer which divides m. Notice that if two positive integers have the same odd part, then one of them divides the other. The numbers $n+1$, $n+2$, $\ldots$, $2n$ have the property that none of them divides any of the others because the ratio of the larger to the smaller is less than 2. Therefore these n numbers have different odd parts k in the range $1 \le k \le 2n-1$. There are exactly n positive odd positive integers less than $2n$, so our odd parts are all the odd positive integers from 1 to $2n-1$, and the sum of those numbers is n^2.

2. Suppose that you have ten different two-digit numbers. Is it necessarily true that one may choose two disjoint non-empty subsets so that their elements have the same sum? *Hint:* $10 \times 99 = 990 < 1024 = 2^{10}$.
   Solution Let S denote our set of 10 different two-digit numbers. For each T a subset of S, let its *sum* be the sum of the integers in the set (with the proviso that the sum of the members of the empty set is 0. A set of size 10 has $2^{10} = 1024$ subsets. Each subset has a digit sum strictly less than $10 \times 99 = 990$ so two different subsets A and B have the same digit sum. Remove the elements of $A \cap B$ from both sets, to yield sets A' and B' respectively. The sets A' and B' must be disjoint (no element in common), they must be different (because A and B were different), and the sums of A' and B' must be equal. Notice that this existence proof does not tell you an efficient way to find A' and B'.

3. Suppose that we have a set S of 15 positive integers x in the range $1 < x \le 2011$. Suppose also that there is no prime number which divides more than one member of S. Prove that S contains a prime number.
   Solution Suppose, for contradiction, that S contains no prime number. For each $n \in S$, let p_n denote the smallest prime divisor of s. Therefore the 15 prime numbers p_n must be different. The fifteenth prime is 47, so at least one of our 15 numbers m is such that $p_m \ge 47$. However, m is not prime and

is divisible by no prime number less than 47. Therefore $m \geq 47^2 = 2209$. This is impossible by the conditions of the problem, so S must contain a prime number.