Olympiad Maths Club - Sheet 4

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These are old BMO1 problems from the easier end of the papers.

1. Let n be an integer greater than 6. Prove that if $n-1$ and $n+1$ are both prime, then $n^2(n^2+16)$ is divisible by 720. Is the converse true?

2. Adrian teaches a class of six pairs of twins. He wishes to set up teams for a quiz, but wants to avoid putting any pair of twins into the same team. Subject to this condition:

   (i) In how many ways can he split them into two teams of six?

   (ii) In how many ways can he split them into three teams of four?

3. Find four prime numbers less than 100 which are factors of $3^{32} - 2^{32}$.

4. In the convex quadrilateral $ABCD$, points M, N lie on the side AB such that $AM = MN = NB$, and points P, Q lie on the side CD such that $CP = PQ = QD$. Prove that

$$\text{Area of } AMCP = \text{Area of } MNPQ = \frac{1}{3} \text{ Area of } ABCD.$$

5. Find the value of

$$\frac{1^4 + 2007^4 + 2008^4}{1^2 + 2007^2 + 2008^2}.$$

6. Find all solutions in positive integers x, y, z to the simultaneous equations

$$x + y - z = 12$$

$$x^2 + y^2 - z^2 = 12.$$

7. Consider a standard 8×8 chessboard consisting of 64 small squares coloured in the usual pattern, so 32 are black and 32 are white. A *zig-zag* path across the board is a collection of eight white squares, one in each row, which meet at their corners. How many zig-zag paths are there?

8. Find all real values of x, y and z such that

$$(x+1)yz = 12, \ (y+1)zx = 4 \text{ and } (z+1)xy = 4.$$