Olympiad Maths Club - Sheet 2

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1. Prove that there are 1 million consecutive positive integers, none of which are prime numbers.

2. Let a, b and c be real numbers such that $|a-b| \geq |c|, |b-c| \geq |a|$ and $|c-a| \geq |b|$. Prove that one of the numbers a, b and c is the sum of the other two.

3. Lines are always deemed to extend infinitely in both directions. Suppose that n different lines are drawn in the plane, dividing the plane into regions (countries).

   (a) What is the maximum number of countries which may arise (in terms of n).

   (b) Prove that you can use 2 colours to colour the regions so that countries sharing a border (not at a point) are coloured differently.

4. Determine all positive integers n such that there are n consecutive positive integers that sum to a square.
   (**Partial Solution already established** If n is odd, then consider the n consecutive positive integers with central term n. The sum of these terms is n times their average, and so is n^2.)

   What happens when n is even? In this case, the easier thing to do is to show that if the largest integer t such that 2^t divides n is even, then no sum of consecutive positive integers can be a square. If you can manage that, then move on to considering the hardest case and try to show that if the largest integer t such that 2^t divides n is odd, then the sum of n consectutive positive integers can be a square.

5. Ten positive integers are written in a row. A *move* consists of selecting three consecutive integers and increasing each of them by 1. By careful use of such moves, is it possible, after a finite number of moves, to arrange that all ten integers are multiples of 4, irrespective of what they were at the outset? *Consider the sum of the numbers in positions 1, 4, 7 and 10 and other related quantities. See how these quantities change with each move.*

6. Determine which triples of positive integers (a, b, c) satisfy

$$(a^3 + b)(a + b^3) = 2^c.$$

Suppose that (a, b, c) is a solution. Notice that $c \geq 1$ Each of a and b must have the same parity (both even or both odd).

(a) If both are even, produce a contradiction by showing that 2^c must have a factor which is odd and bigger than 1.

(b) Show that if $a = b$, then there is exactly one solution.

(c) Having done (a) and(b), you may assume that $a > b$ and both numbers are odd. Now it gets harder. In fact there is exactly one solution of this form. Can you find it? The hardest thing is to show that this is the unique solution of this form.