

# Olympiad Maths Club Sheet 1

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## Entertainments

1. Consider the first  $n$  positive integers. They are to be arranged in a list in such a way that if  $i, j$  and  $k$  are in the list, and  $i < j < k$ , then it is not the case that  $k$  lies between  $i$  and  $j$  in the list. How many such lists are there? *Here is an example when  $n = 6$ : 6, 4, 1, 2, 3, 5.*

**Solution** First write down 1. Then decide if 2 goes to the left or the right of 1. This gives 2 choices. Now 3 goes to the left or the right of all numbers previously written (and not between them). There are therefore  $2^{n-1}$  possible sequences.

2. Determine all positive integers  $n$  such that there are  $n$  consecutive positive integers that sum to a square.

**Solution** If  $n$  is odd, then consider the  $n$  consecutive positive integers with central term  $n$ . The sum of these terms is  $n$  times their average, and so is  $n^2$ .

Next suppose that  $n = 2m$  is even, consider the consecutive integers

$$k - (m - 1), k - (m - 2), \dots, k + m$$

where  $k$  is an undetermined integer. The sum of these  $2m$  integers is  $2m \times \frac{2k+1}{2} = m(2k+1)$ . Provided that  $m$  is odd, we may define  $k$  by  $2k+1 = mt^2$  where  $t$  is a large odd positive integer, and then the sum of these consecutive integers is  $(tm)^2$ . The role of  $t$  here is to arrange that  $k - (m - 1)$  is positive.

If  $m = 4^a m_0$  where  $m_0$  is odd, we may recycle the last idea and define  $k$  by  $2k+1 = m_0 t^2$ , and the sum of the consecutive integers is  $(2^a m_0 t)^2$ , a square.

If, on the other hand, 2 divides  $m$  an odd number of times, then the same is true  $m(2k+1)$  which therefore cannot be a square.

In summary, for  $n$  even, the sum of  $n$  consecutive positive integers can be a square if, and only if, 2 divides  $n$  an odd number of times.

3. Ten positive integers are written in a row. A *move* consists of selecting three consecutive integers and increasing each of them by 1. By careful use of such moves, is it possible, after a finite number of moves, to arrange that all ten integers are multiples of 4, irrespective of what they were at the outset?

**Solution** Consider three quantities:  $Q_1$  which is the sum of the numbers in positions 1, 4, 7 and 10;  $Q_2$  which is the sum of the numbers in positions 2, 5 and 8;  $Q_3$  which is

the sum of the numbers in positions 3,6 and 9. A move increments each of these three numbers by 1. Therefore  $(1, 0, 0, 0, 0, 0, 0, 0, 0)$  has  $(Q_1, Q_2, Q_3) = (1, 0, 0)$ . Any move adds 1 in all three positions. Therefore after any sequence of moves, at least one entry must be odd and so not divisible by 4. If this happens, at least one of the ten numbers must be not divisible by 4.

4. Ten thirsty drinkers can collectively drink a large pot dry in one minute. Each drinks at a constant rate, but that rate may vary from drinker to drinker. When refilled, this pot can be decanted to fill 10 identical bottles exactly. Can the thirsty drinkers arrange to drink all the bottles dry in one minute? The drinkers can swap bottles around during the minute, and the process of switching bottles is deemed to take no time. Each drinker can drink from a bottle at exactly the same rate that they drink from the pot.

**Solution** Here is a strategy which works. The 10 drinkers sit in a circle, and pass a bottle to their right every 6 seconds. After one minute, every bottle has received the same treatment and they are all empty.

However, there is something dodgy going on. How do we know that some bottles are not finished early? Start again, and instead of passing bottles, pass round 10 large pots using the 6 second rule. No pot can be finished early because even the fastest drinker needs at least one minute to drink a pot dry, and each pot is drunk from at rates which are at most that of the fastest drinker. Each pot has been similarly treated (6 seconds from each drinker) so at the end of the minute, a total of one potful has been drunk (for that is how much the drinkers can consume in one minute) so each of the pots has an amount of liquid missing which is exactly enough to fill one bottle. Replace the 10 pots by 10 bottles and you are done.

5. Determine which triples of positive integers  $(a, b, c)$  satisfy

$$(a^3 + b)(a + b^3) = 2^c.$$

**Solution** Notice that  $c \geq 1$  in any solution. Each of  $a$  and  $b$  must have the same parity (both even or both odd). If both are even, you get into trouble:  $a = 2^s m$  and  $b = 2^t n$  with  $m, n$  odd and  $s, t \geq 1$ . Then  $3s = t$  and  $3t = s$  (or you produce an odd factor bigger than 1 of  $2^c$ ). However  $3s = t$  and  $3t = s$  have no solutions among positive integers. Therefore  $a$  and  $b$  are odd.

If  $a = b$  the only solution is  $a = b = 1$ . Without loss of generality we may assume that  $a > b$ , so  $a^3 + b > b^3 + a$  because

$$a^3 + b - b^3 - a = (a - b)(a^2 + ab + b^2) - (a - b) = (a - b)(a^2 + ab + b^2 - 1) > 0.$$

Now  $b^3 + a$  divides  $a^3 + b$  because they are both powers of 2. However  $b^3 + a$  divides  $b^9 + a^3$  because  $b^9 + a^3 = (b^3 + a)(b^6 - ab^3 + a^2)$ . Therefore  $b^3 + a$  divides  $(b^9 + a^3) - (a^3 + b)$  which is  $b^9 - b$ . Now  $b^9 - b = b(b^2 - 1)(b^2 + 1)(b^4 + 1)$ . The last two factors are both 2 more than a multiple of 4 (why?), and so are each divisible by 2 but not by 4.

Therefore  $b^3 + a$  (which is a power of 2) divides  $4b(b^2 - 1)$ . Therefore  $b^3 < 4(b^2 - 1)$  so  $b \leq 3$ .

If  $b = 1$  then  $a^3 + 1$  and  $a + 1$  are both powers of 2, so  $a^2 - a + 1$  is a power of 2 which is false unless  $a = 0$  for parity reasons. However,  $a > 0$ . On the other hand, if  $b = 3$ , then  $b^3 + a$  divides  $4(b^2 - 1)$  so  $27 + a$  divides 32 so  $a = 5$ . Thus we have candidate solutions  $(1, 1, 2)$ ,  $(3, 5, 12)$ ,  $(5, 3, 12)$  and by inspection, these are all in fact solutions.

Note the logic at the end. Our mathematics showed that if  $(a, b, c)$  was a solution, then  $(a, b, c)$  had to be one of those three possibilities. However, there is the concern that not all of them (possibly none of them) are actually solutions. However, it is easy to verify that in fact these are all solutions.