Circles in areals

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Introduction

August Möbius introduced the system of barycentric or areal co-ordinates in 1827[1, 2]. The idea is that one may attach weights to points, and that a system of weights determines a centre of mass. Given a triangle ABC, one obtains a co-ordinate system for the plane by placing weights x, y and z at the vertices (with $x + y + z = 1$) to describe the point which is the centre of mass. The vertices have co-ordinates $A = (1, 0, 0)$, $B = (0, 1, 0)$ and $C = (0, 0, 1)$. By scaling so that triangle ABC has area $[ABC] = 1$, we can take the co-ordinates of P to be $([PBC], [PCA], [PAB])$, provided that we take area to be signed. Our convention is that anticlockwise triangles have positive area.

Points inside the triangle have strictly positive co-ordinates, and points outside the triangle must have at least one negative co-ordinate (we are allowed negative masses). The equation of a line looks like the equation of a plane in Cartesian co-ordinates, but note that the equation $lx + ly + lz = 0$ (with $l \neq 0$) is not satisfied by any point (x, y, z) of the Euclidean plane. We use such equations to describe the line at infinity when doing projective geometry.

The equation of a circle looks rather different to its Cartesian cousin. Let a, b and c be the lengths of the sides of triangle ABC which has circumcircle with centre O and radius R. The expression $a^2yz + b^2zx + c^2xy$ has several interpretations. In the language of mechanics it is the moment of inertia of the masses x, y and z placed at A, B and C. Equally well, in the language of statistics, it is the variance of this weighted system of points, and for this reason we introduce σ^2 as shorthand for $a^2yz + b^2zx + c^2xy$. A third interpretation comes from the fact that

$$\sigma^2 = R^2 - OP^2 \tag{1}$$

is minus the power of $P = (x, y, z)$ with respect to the circumcircle. This is because of the parallel axis theorem of mechanics, also known as the Huygens-Steiner theorem [3, 4]. We can calculate the mean of the square of the distances from O to the weighted points A, B and C in two ways. This quantity is clearly R^2 since the weights are irrelevant. However, by the Huygens-Steiner theorem it is also $OP^2 + \sigma^2$.

Thus the equation of the circumcircle is $\sigma^2 = 0$, and $\sigma^2(x, y, z)$ is positive or negative as $P = (x, y, z)$ is inside or outside the circumcircle. It is the existence of negative weights which may cause σ^2 to assume a negative value.

The corresponding minus the power function for a different circle Γ is $\sigma^2+lx+my+nz$. This is because when you subtract the power functions of two different circles you obtain the equation of their radical axis, and that is a line. In the event that the circles are concentric, the radical axis is the line at infinity. The family of concentric circles with centre O therefore have minus the power functions $\sigma^2 + l(x + y + z)$ as l varies. Thus the situation is very similar to the one pertaining in Cartesian co-ordinates, where the unit circle has power function $x^2 + y^2 - 1$, and a general circle is obtained by adding linear terms (and constants).

Returning to the areal circle minus the power function $\sigma^2+lx+my+nz$, we see that l is minus the power of A with respect to Γ by evaluating at $(1, 0, 0)$, with similar interpretations for m and n. This allows us to write down the equations, or minus the power functions, of circles with respect to which the powers of A, B and C can be determined. This includes the incircle, the excircles, nine-point circle and the circles with diameters such as II_a. Here I is the incentre and I_a is an excentre of ABC. Other circles which are readily accessible in this fashion include the circles on diameters AB, BC and CA, the circles on diameters OA, OB and OC, and the circles on diameters HA, HB and HC. Also the circle which passes through A and is tangent to the line BC at B, and its five siblings are all accessible, and these circles can be used (via radical axes) to calculate the areal co-ordinates of the two Brocard points of triangle ABC.

However, some important circles associated with triangle geometry are not accessible by the method just described. For example, the orthocentroidal circle on diameter GH (centroid and orthocentre), the Fuhrmann circle on diameter HNa (orthocentre and Nagel point) and the Brocard circle on diameter OK (circumcentre and symmedian point) on which both Brocard points lie[5]. These circles seem to be hard to access by synthetic Euclidean means, and having their power functions immediately available in areal standard form should make them more tractable. We present a method which allows us to obtain areal descriptions of minus the power functions of these circles. The method will work in higher dimensions to render certain spheres and hyperspheres amenable to barycentric study.

There is an areal formula for the inner product of two vectors (a vector being the difference of two points). This can be used to obtain the equation of the circle on diameter UV as follows. Let $P = (x, y, z)$ be a variable point on the circle. Let $U = (u_1, v_1, w_1)$ and $V = (u_2, v_2, w_2)$. Using the areal scalar product formula, the areal condition for $\vec{UP}$ and $\vec{VP}$ to be orthogonal is that

$$a^2((y-v_1)(z-w_2)+(z-w_1)(y-v_2))+b^2((z-w_1)(x-u_2)+(x-u_1)(z-w_2))$$
$$+c^2((x-u_1)(y-v_2)+(y-v_1)(x-u_2))=0$$

which may not be a convenient condition to use.

The Result

We present the result in terms of areal co-ordinates in the plane, but it holds in more generality using arbitrary weighted systems of points in Euclidean space, and weighted combinations of power functions or minus the power functions.

Theorem Let X and $Y = (u, v, w)$ be points in the plane of ABC. Let σ_a^2, σ_b^2 and σ_c^2 be minus the power functions of the circles on diameters XA, XB and XC respectively. It follows that $u\sigma_a^2 + v\sigma_b^2 + w\sigma_c^2$ is minus the power function of the circle on diameter XY.

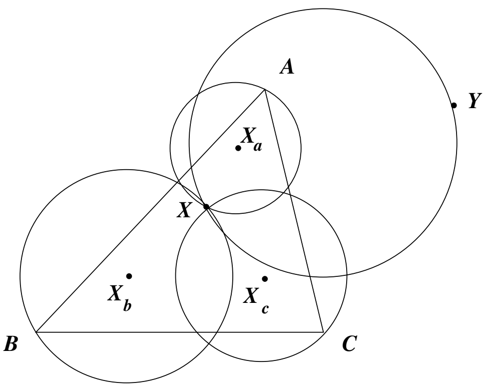

Figure 1: Weighting circles

Proof The centres of the circles on diameters XA, XB and XC are denoted X_a, X_b and X_c respectively. By taking a linear combination of minus the power functions we see that

$$u\sigma_a^2 + v\sigma_b^2 + w\sigma_c^2 = k_1 - uPX_a^2 - vPX_b^2 - wPX_c^2 \quad (2)$$

where $P = (x, y, z)$ is a point of the plane and $k_1 = uAX_a^2 + vBX_b^2 + wCX_c^2$.

Now can again deploy the Huygens-Steiner theorem just as we did to obtain equation (1). This time we obtain

$$uPX_a^2 + vPX_b^2 + wPX_c^2 = PZ^2 + k_2, \quad (3)$$

where Z is the centre of mass of the weighted points uX_a, vX_b, wX_c and k_2 is independent of P.

Triangle $X_aX_bX_c$ is a triangle similar to ABC, and is the result of enlarging ABC from X with scale factor $1/2$. The centre of mass of uA, vB, wC is Y so the centre of mass of uX_a, vX_b, wX_c is Z where Z is the midpoint of XY. Using equations (2) and (3) we find that

$$u\sigma_a^2 + v\sigma_b^2 + w\sigma_c^2 = k_1 - k_2 - PZ^2 = k_3 - PZ^2,$$

where k_3 is independent of P. Therefore $u\sigma_a^2 + v\sigma_b^2 + w\sigma_c^2$ is minus the power function of a circle with centre Z which passes through X (since $\sigma_a^2(X) = \sigma_b^2(X) = \sigma_c^2(X) = 0$). This must therefore be the circle on diameter XY.

Famous Circles

Using the circumcentre as X

An attractive choice for X is O, the circumcentre of ABC. This is because the circles on diameters such as OA pass through the midpoints of the sides, and so have straightforward minus the power functions. Moving X to O and borrowing the earlier notation,

$$\begin{aligned}
\sigma_a^2 &= \sigma^2 - \frac{1}{2}c^2y - \frac{1}{2}b^2z; \\
\sigma_b^2 &= \sigma^2 - \frac{1}{2}c^2x - \frac{1}{2}a^2z; \\
\sigma_c^2 &= \sigma^2 - \frac{1}{2}b^2x - \frac{1}{2}a^2y.
\end{aligned}$$

For example, we calculate the coefficient of y in the first expression by noting that the power of B with respect to the relevant circle is $\frac{1}{2}AB \cdot AB = \frac{1}{2}c^2$.

The centroid, incircle and symmedian points of ABC have unnormalized areal co-ordinates $(1,1,1), (a,b,c)$ and (a^2,b^2,c^2) respectively. We normalize these co-ordinates and deploy the theorem: the circle on diameter OG therefore has minus the power function

$$\sigma^2 - \frac{1}{6}(b^2+c^2)x - \frac{1}{6}(c^2+a^2)y - \frac{1}{6}(a^2+b^2)z,$$

the circle on diameter OI has minus the power function

$$\sigma^2 - \frac{1}{a+b+c}\left(\frac{1}{2}(bc^2+b^2c)x + \frac{1}{2}(ca^2+c^2a)y + \frac{1}{2}(ab^2+a^2b)z\right)$$

and the minus the power function of the Brocard circle on diameter OK is

$$\sigma^2 - \frac{b^2c^2}{a^2+b^2+c^2}x - \frac{c^2a^2}{a^2+b^2+c^2}y - \frac{a^2b^2}{a^2+b^2+c^2}z.$$

Using the orthocentre as X

We use H rather than O for X in the theorem, and look for minus the power functions of other interesting circles. We redefine σ_a^2, σ_b^2 and σ_c^2 for the new value of X. We obtain

$$\begin{aligned}\sigma_a^2 &= \sigma^2 - ca\cos By - ab\cos Cz;\\ \sigma_b^2 &= \sigma^2 - ab\cos Cz - bc\cos Ax;\\ \sigma_c^2 &= \sigma^2 - bc\cos Ax - ca\cos By.\end{aligned}$$

We repeat our method to discover that the orthocentroidal circle on diameter HG has minus the power function

$$\sigma^2 - \frac{2}{3}bc\cos Ax - \frac{2}{3}ca\cos By - \frac{2}{3}ab\cos Cz, \qquad (4)$$

the circle on diameter HI has minus the power function

$$\sigma^2 - \frac{bc(b+c)\cos A}{a+b+c}x - \frac{ca(c+a)\cos B}{a+b+c}y - \frac{ab(a+b)\cos C}{a+b+c}z,$$

and minus the power function of the circle on diameter HK is

$$\sigma^2 - \frac{bc(b^2+c^2)\cos A}{a^2+b^2+c^2}x - \frac{ca(c^2+a^2)\cos B}{a^2+b^2+c^2}y - \frac{ab(a^2+b^2)\cos C}{a^2+b^2+c^2}z.$$

Another circle of interest is the Fuhrmann circle, defined as the circle with diameter HNa, where Na is the Nagel point, the point of concurrency of the Cevians which join each vertex of ABC to the opposite contact point of an excircle. Now $Na = (\frac{s-a}{s}, \frac{s-b}{s}, \frac{s-c}{s})$ so the coefficient of x in the minus the power function of the Fuhrmann circle is

$$-\frac{s-b}{s}bc\cos A - \frac{s-c}{s}bc\cos A = -\frac{2abc}{(a+b+c)}\cos A = -4Rr\cos A.$$

Therefore minus the power function of the Fuhrmann circle is

$$\sigma^2 - 4Rr(\cos Ax + \cos By + \cos Cz).$$

Radical Axes

The most algebraically intriguing radical axes which are now available are (i) that for the circles on diameters OI and HI, and (ii) that for the circles on diameters OK and HK.

In case (i), subtraction reveals that the radical axis has equation

$$bc(b+c)(1-2\cos A)x + ca(c+a)(1-2\cos B)y + ab(a+b)(1-2\cos C)z,$$

a line through the incentre of ABC.

In case (ii), subtraction reveals that the radical axis has equation

$$(a^2b^2 + a^2c^2 - b^4 - c^4)x + (b^2c^2 + a^2b^2 - c^4 - a^4)y + (c^2a^2 + b^2c^2 - a^4 - b^4)z,$$

a line through the symmedian point of ABC.

Circles on arbitrary diameters

Let σ^2_{XY} denote the minus the power function of the circle on diameter XY. Notice that $\sigma^2_{XY} = \sigma^2_{YX}$ and in this context we allow degenerate circles on diameters such as XX.

We apply the theorem when X is at a vertex and $Y = (x_1, y_1, z_1)$. Then

$$\sigma^2_{AY} = x_1\sigma^2_{AA} + y_1\sigma^2_{AB} + z_1\sigma^2_{AC};$$

$$\sigma^2_{BY} = x_1\sigma^2_{BA} + y_1\sigma^2_{BB} + z_1\sigma^2_{BC};$$

$$\sigma^2_{CY} = x_1\sigma^2_{CA} + y_1\sigma^2_{CB} + z_1\sigma^2_{CC}.$$

Having done this, we are now in a position to apply the theorem once more, with our current Y playing the role of X in the theorem, and a new $Y = (x_2, y_2, z_2)$ playing the role of X. Then

$$\sigma^2_{XY} = x_2\sigma^2_{AY} + y_2\sigma^2_{BY} + z_2\sigma^2_{CY}.$$

This yields the symmetric bilinear expression

$$\sigma^2_{XY} = (x_2, y_2, z_2)\begin{pmatrix} \sigma^2_{AA} & \sigma^2_{AB} & \sigma^2_{AC} \\ \sigma^2_{BA} & \sigma^2_{BB} & \sigma^2_{BC} \\ \sigma^2_{CA} & \sigma^2_{CB} & \sigma^2_{CC} \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}.$$

Write $\sigma^2_{UV} = \sigma^2 + \lambda_{UV}$, then using bilinearity we obtain

$$\sigma^2_{XY} = \sigma^2 + (x_2, y_2, z_2)\begin{pmatrix} \lambda_{AA} & \lambda_{AB} & \lambda_{AC} \\ \lambda_{BA} & \lambda_{BB} & \lambda_{BC} \\ \lambda_{CA} & \lambda_{CB} & \lambda_{CC} \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}.$$

Observe that $\lambda_{AA} = -c^2y - b^2z$ and $\lambda_{AB} = \lambda_{BA} = -ab\cos Cz = \frac{1}{2}(c^2 - a^2 - b^2)z$ with other quantities obtained by cyclic change of letters.

It is much easier to write down minus the power function of the circle on diameter XY if the co-ordinates of both X and Y are obtained by cyclic change of letters, because then it suffices to work with the coefficient of x, and later generate those of y and z by cyclic change of letters. The coefficient of x is

$$(x_2, y_2, z_2)\begin{pmatrix} 0 & 0 & 0 \\ 0 & -c^2 & -bc\cos A \\ 0 & -bc\cos A & -b^2 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}.$$

For example, one might be investigating the orthocentroidal circle, using $G = (1/3, 1/3, 1/3)$ and $H = (\cot B\cot C, \cot C\cot A, \cot A\cot B)$. The result, after manipulation, is $-\frac{2}{3}bc\cos A$ and we recover (4).

Afterthoughts

The inspiration for this method came from the observation that the areal equation of a line is $lx+my+nz = 0$, where the perpendicular directed distances from A, B and C to the line are in the ratio $l : m : n$, so we can view this equation as a barycentric combination of the sides lines $x = 0$, $y = 0$ and $z = 0$ of triangle ABC. Thus points and lines are accessible as barycentric combinations, so why not circles?

The incentre of triangle ABC is interior to the orthocentroidal circle [6], and in fact can arise at any point interior to that disk [7], [4] in the sense that given the locations of G, H and I with the latter point interior to the circle on diameter GH, it is possible to deduce unique positions for A, B and C which give rise to this configuration. The fact that I is trapped in this disk amounts to Euler's inequality that $2r < R$ where r is inradius of ABC. We will need to use Heron's formula that $16[ABC]^2 = (a + b + c)(a + b - c)(b + c - a)(c + a - b)$ and the facts that $[ABC] = abc/4R = rs$ and that $R^2 - OI^2 = abc/(a + b + c)(= 2Rr)$. Substitute the areal co-ordinates of the incentre into the orthocentroidal minus the power function (4) to get

$$\frac{abc}{2s} + \frac{1}{6s}\left(a^3 + b^3 + c^3 - ab^2 - a^2b - bc^2 - b^2c - ca^2 - c^2a\right)$$

since

$$\sigma^2\left(\frac{a}{2s}, \frac{b}{2s}, \frac{c}{2s}\right) = \frac{abc}{a + b + c}.$$

The value is therefore

$$\frac{1}{6s}\left((a - b - c)(c - b - a)(c - a - b) + abc\right)$$

$$= \frac{1}{6s}\left(4R[ABC] - \frac{16[ABC]^2}{2s}\right)$$

$$= \frac{1}{6s}\left(4R[ABC] - 8[ABC]r\right) = \frac{4[ABC]}{6s}(R - 2r) > 0.$$

The symmedian point K of triangle ABC is always in the interior of the orthocentroidal circle [8]. This can now be established by simply substituting into the appropriate minus the power function. This time, the fact that K is trapped in this disk is equivalent to $\frac{a^2+b^2+c^2}{9} < R^2$, which is to say $OG^2 > 0$.

Let $\Sigma = a^2 + b^2 + c^2$, then substituting $(a^2/\Sigma, b^2/\Sigma, c^2/\Sigma)$ into minus the orthocentroidal power function we obtain

$$\frac{1}{\Sigma^2}(3a^2b^2c^2) - \frac{16}{3\Sigma}[ABC].$$

This minus the power function evaluated at the symmedian point is

$$\frac{48[ABC]^2}{\Sigma^2}\left(R^2-\frac{a^2+b^2+c^2}{9}\right)=\frac{48[ABC]^2}{\Sigma^2}\left(R^2-\sigma^2\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)\right)=\frac{48[ABC]^2}{\Sigma^2}OG^2>0.$$

The expression $\sigma^2+lx+ny+nz$ is not in homogeneous form. It is common to see this written as $\sigma^2+(lx+ny+nz)(x+y+z)$, because $x+y+z=1$ and the expression becomes homogeneous of degree 2. One may then substitute in unnormalized areal co-ordinates and draw meaningful conclusions. This can be very convenient from the point of view of calculation, but it is an act of algebraic hooliganism to expand the final product since the meanings of l, m and n are lost.

Bradley's accessible text [9] is a modern comprehensive reference for areal co-ordinates as a special case of generalized homogeneous co-ordinates.

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