Group Theory: Sheet 8

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1. You have 10 beads, 6 are black and 4 are white. How many essentially different necklaces can you make?

2. (repeat) You have 24 beads to thread on to a circular string to make a necklace. There are 8 blue beads, 6 red beads, 4 white beads, 2 black beads and 4 green beads. How many essentially different necklaces can you make? (You may need a calculator, or if that is too dreadful a thought, you can give a formula).

3. (repeat) Let G be a finite group. It turns out that the average (over all group elements) size of the centralizer of an element is the number of conjugacy classes of the group. Why?

4. Suppose that G is a group of order 1,000. Show that G must have a normal subgroup N with $1 \neq N \neq G$.

5. Let H be a subgroup of finite index in a group G.

   (a) Let $\Omega = \{H^x \mid x \in G\}$. Let G act on Ω by conjugation. Deduce that Ω is finite.

   (b) Let
   $$K = \bigcap_{g \in G} H^g.$$
   Show that K is a normal subgroup of G and that $|G : K| < \infty$.

6. Let G be a finite group, p be a prime number and suppose that $P \in \mathrm{Syl}_p(G)$.

   (a) Show that $P \cap N \in \mathrm{Syl}_p(G)$.

   (b) Show that $PN/N \in \mathrm{Syl}_p(G/N)$.