Group Theory: Sheet 6

The course web site is available via `http://www.bath.ac.uk/~masgcs/`

1. Let G be a group and let G act on itself by conjugation. Choose a transversal $\{x_i \mid i \in I\}$ for the conjugacy classes.

   (a) Suppose that G is finite. Prove that
   $$|G| = \sum_{i \in I} |G : C_G(x_i)|$$
   by applying the orbit-stabilizer theorem.

   (b) Suppose that G is finite of order p^n where $n \geq 1$, Use part (a) to show that $|Z(G)| \equiv 0 \bmod p$ and deduce that $|Z(G)| > 1$.

2. Let G be a group of order p^2 where p is a prime number.

   (a) Show that if G is non-abelian, we must have $|Z(G)| = p$ and $g^p = 1$ for every $g \in G$.

   (b) Suppose that G is non-abelian, and choose $h \in G$ with $h \notin Z(G)$. Let $H = \langle h \rangle$. Show that $H \leq N_G(H)$ and $Z(G) \leq N_G(H)$, and deduce that $H \trianglelefteq G$.

   (c) Prove that $Z(G)H = G$ and $Z(G) \cap H = 1$. Deduce that every group of order p^2 is abelian.

3. Let G be a finite group of even order. Show that the number of elements which do not have order 2 is even, and deduce that G contains an element of order 2.

4. Let G be a finite group and suppose that p is a prime number which divides $|G|$. Let
   $$\Omega = \{(g_1, g_2, \ldots, g_p) \mid g_1 g_2 \ldots g_p = 1\}.$$

   (a) Prove that $|\Omega| = |G|^{p-1}$.

   (b) Let $H = \langle x \rangle$ be a cyclic group of order p. Define a multiplication by $(g_1, g_2, \ldots, g_p) \cdot x^i = (g_{1+i}, g_{2+i}, \ldots, g_{p+i})$ for each $(g_1, g_2, \ldots, g_p) \in \Omega$. Show that every product $(g_1, g_2, \ldots, g_p) \cdot x^i$ is in Ω and demonstrate that this multiplication gives an action of H on Ω.

(c) Show that each orbit of H acting on Ω has size 1 or p.

(d) Describe the orbits of H on Ω which have size 1.

(f) Use the orbit-stabilizer theorem to prove Cauchy's result that if a prime number p divides the order of a finite group, then there is $g \in G$ with g of order p (i.e. $|\langle g \rangle| = p$).

(g) How does Question 4(f) relate to Question 3?

5. Suppose that $H \leq G$. Show that the number of conjugates subgroups of the form H^g (as g ranges over G) is a divisor of $|G|$.

6. Suppose that G is a group and that H is a subgroup of G. If $g \in N_G(H)$, show that $\tau_g : H \to H$ defined by $h \mapsto h^g$ for every $h \in H$ is an automorphism of H. Show that $\tau : g \mapsto \tau_g$ is a homomorphism from $N_G(H)$ to Aut (H). Deduce that $C_G(H) \trianglelefteq N_G(H)$ and that $N_G(H)/C_G(H)$ is isomorphic to a subgroup of Aut (H).