Group Theory: Sheet 5

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1. Let G be a group, and let S be a subset of G. Let

   $$C_G(S) = \{g \in G \mid g^{-1}sg = s \forall s \in G\} \text{ and } N_G(S) = \{g \in G \mid g^{-1}Sg = S\}.$$

   (a) Prove that $C_G(S) \leq N_G(S) \leq G$.

   (b) Prove that $C_G(S) \trianglelefteq N_G(S)$.

   (c) Show that $N_G(S)/C_G(S)$ is isomorphic to a subgroup of Sym$S$.

2. Let G be a finitely generated group in which for each $x \in G$, the set $S_x = \{g^{-1}xg \mid g \in G\}$ is finite. Let $Z(G)$ denote the centre of G (the subgroup of G consisting of those elements of G which commute with all elements of G). Prove that $Z(G)$ has finite index in G. *(Hint: Do Question 1 first)*

3. Prove that Aut S_5 is isomorphic to S_5.

4. Suppose that $G = \langle x, y \rangle$ and $x^2 = y^2 = 1$. Prove that G has a cyclic subgroup of index at most 2.

5. Suppose that $G = \langle S \rangle$ is a group and that S is finite. Now suppose that also $G = \langle T \rangle$. Show that there is a finite subset U of T such that $G = \langle U \rangle$.

6. Consider the maps $\alpha, \beta \in$ Sym $\mathbb{Z}$ defines by $\alpha : i \mapsto i + 2$ for all $i \in \mathbb{Z}$, and $\beta = (0, 1)$. Thus β fixes all integers except 0 and 1, and it swaps those. Consider the groups $G = \langle \alpha, \beta \rangle$ and $H = \langle T \rangle$ where $T = \{\alpha^{-k}\beta\alpha^k \mid k \in \mathbb{Z}\}$. Observe that all elements of H have finite support (i.e. each one moves only finitely many integers). Deduce that a finitely generated group may have a subgroup which has no finite generating set.

7. Let G be a finite cyclic group of order n. Suppose that d is a divisor of n. Show that G has a unique subgroup of order d.