Group Theory: Sheet 4

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1. Let G be a group.
   (a) Suppose that $H \leq G$ and $|G : H| = 2$. Prove that $H \trianglelefteq G$.
   (b) Show that S_3 has a non-normal subgroup of index 3.

2. Let G be a group, and suppose that H, K are subgroups of G.
   (a) Show that HK is a subgroup of G if and only if $HK = KH$.
   (b) Suppose that $K \trianglelefteq G$. Show that HK is a subgroup of G.
   (c) Suppose that G is a finite group. Show that $|HK| \cdot |H \cap K| = |H| \cdot |K|$.

3. Suppose that G is a group and that N is a finite normal subgoup of G. Show that G must have a subgroup H of finite index with the property that every element of H commutes with every element of N. *Hint: for each $g \in G$ there $\sigma_g \in Aut(N)$ defined by $n \mapsto g^{-1}ng$ for every $n \in N$.*

4. Let G be a group.
   (a) Show that $x, y \in G$ commute if and only if $x^{-1}y^{-1}xy = 1$.
   (b) Suppose that M, N are normal subgroups of G such that $M \cap N = 1$. Show that each element of M commutes with each element of N.
   (c) Suppose that M, N are normal subgroups of G such that both G/M and G/N are abelian. Prove that G is an abelian group.

5. Show that $\mathrm{Aut}(S_4)$ is isomorphic to S_4.

6. (a) Suppose that G is a group. Show that the map $i : G \to G$ defined by $g \mapsto g^{-1}$ for every $g \in G$ is an automorphism of G if and only if G is abelian.
   (b) Let G be a group with the property that $g^2 = 1$ for every $g \in G$. Prove that G must be abelian, and show that G (written additively) may be regarded as a vector space over the field $\mathbb{F}_2$ with two elements.
   (c) Suppose that G is a finite group with a trivial automorphism group. Prove that G has either 1 or 2 elements.