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A Triangle and its Image under a Half Turn

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Abstract A triangle ABC and its image DEF under a half turn results in circles BCD, CAE, ABF intersecting at a point P on circle DEF. A point Q on circle ABC is similarly defined. Also it is established that a conic passes through A, B, C, D, E, F, P, Q

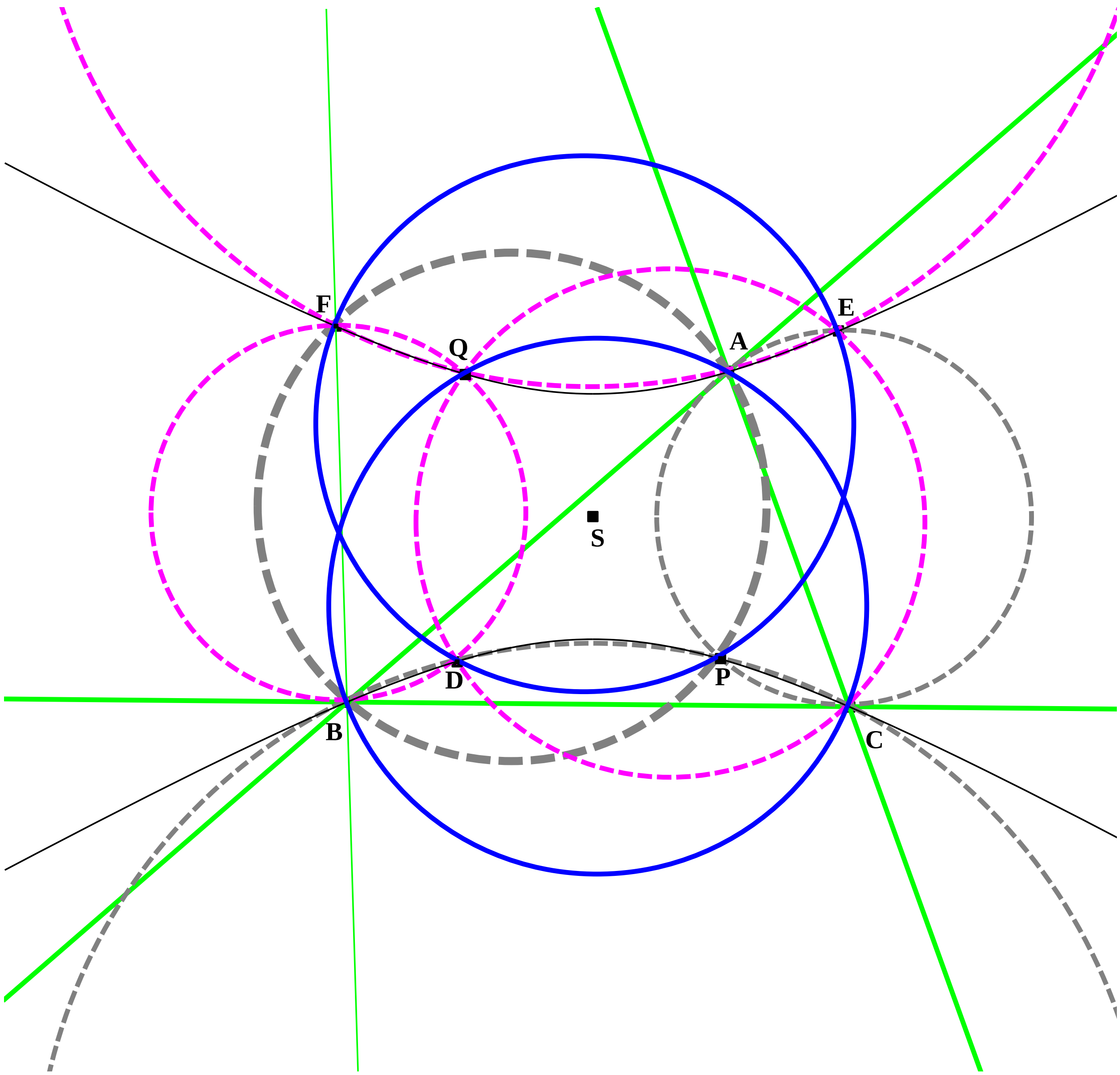

Fig. 1
A Triangle ABC its image DEF and intersecting circles such as ABF, BCD, CAE

1. Introduction

Given triangle ABC the points D, E, F are the images of A, B, C respectively under a half-turn rotation about an arbitrary point S. It is found that circles ABF, BCD, CAE have a common point P that also lies on the circumcircle of triangle DEF. Similarly (in fact, by symmetry) circles DEC, EFA, FDB have a common point Q lying on the circumcircle of triangle ABC.

We also prove that A, B, C, D, E, F, P, Q lie on a conic. Areal co-ordinates are used throughout with ABC as triangle of reference.

2. The points D, E, F and circles BCD, CAE, ABF

We suppose that S has co-ordinates (l, m, n). Then the co-ordinates of D, E, F are as follows:

$$D(2l - 1, 2m, 2n), E(2l, 2m - 1, 2n), F(2l, 2m, 2n - 1). \qquad (2.1)$$

Here it is important to realize expressions (2.1) are correct if, and only if,

$$l + m + n = 1. \qquad (2.2)$$

The equation of the circle BCD may now be worked and is best stated in the form

$$a^2yz + b^2zx + c^2xy + \{2/(1 - 2l)\}(2a^2mn + (2l - 1)(b^2n + c^2m))x(x + y + z) = 0. \qquad (2.3)$$

The equations of the circles CAE and ABF follow from Equation (2.3) by cyclic change of x, y, z and a, b, c and l, m, n.

The co-ordinates of the common point P of the three circles are very involved, but may easily be checked using an algebra computer software package.

3. The conic ABCDEF

If there is a conic through these six points then since it passes through A, B, C it must have an equation of the form

$$fyz + gzx + hxy = 0, \qquad (3.1)$$

for some constants f, g, h. Substituting in the co-ordinates of D and E we find the conic ABCDE has equation

$$l(1 - 2l)yz + m(1 - 2m)zx + n(1 - 2n)xy = 0. \qquad (3.2)$$

Its symmetric form shows that it also passes through F.

4. The point Q

The equation of circle ABC is of course

$$a^2yz + b^2zx + c^2xy = 0. \qquad (4.1)$$

It meets the conic ABCDEF at the point Q with co-ordinates (x, y, z), where

$$x = 1/(b^2n(2n – 1) – c^2m(2m – 1)),$$
$$y = 1/(c^2l(2l – 1) – a^2n(2n – 1)), \qquad (4.2)$$
$$z = 1/(a^2m(2m – 1) – b^2l(2l – 1)).$$

If S is at the centroid, then this point is, of course, the Steiner point.

5. The circle DEC

The equation of circle DEC is

$$2(b^2n(2l + 2m – 1) + c^2m(2m – 1))x^2 + 2(a^2n(2l + 2m – 1) + c^2l(2l – 1))y^2$$
$$– (a^2(2l + 2m – 1)^2 + 2c^2l(1 – 2l))yz – (b^2(2l + 2m – 1)^2 + 2c^2m(1 – 2m))zx$$
$$+ (2a^2n(2l + 2m – 1) + 2b^2n(2l + 2m – 1) – (2l(4m + 2n – 1) + (2n – 1)(2m – 1)))c^2xy = 0.$$
$$(5.1)$$

The equations of circle EFA, FDB may now be written down by cyclic change of x, y, z and a, b, c and l, m, n.

It may now be verified that Q lies on these circles. By symmetry it follows that P lies on the conic and on the circles BCD, CAE, ABF, and DEF.

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