Article: CJB/97/2010

How the Excentres create Points on the Circumcircle

Christopher Bradley

Abstract: The circles through pairs of vertices of a triangle and the excentres opposite the third vertex have centres lying on the circumcircle and pass through the incentre of the triangle. The triangles with these centres as vertices exhibit properties that are described.

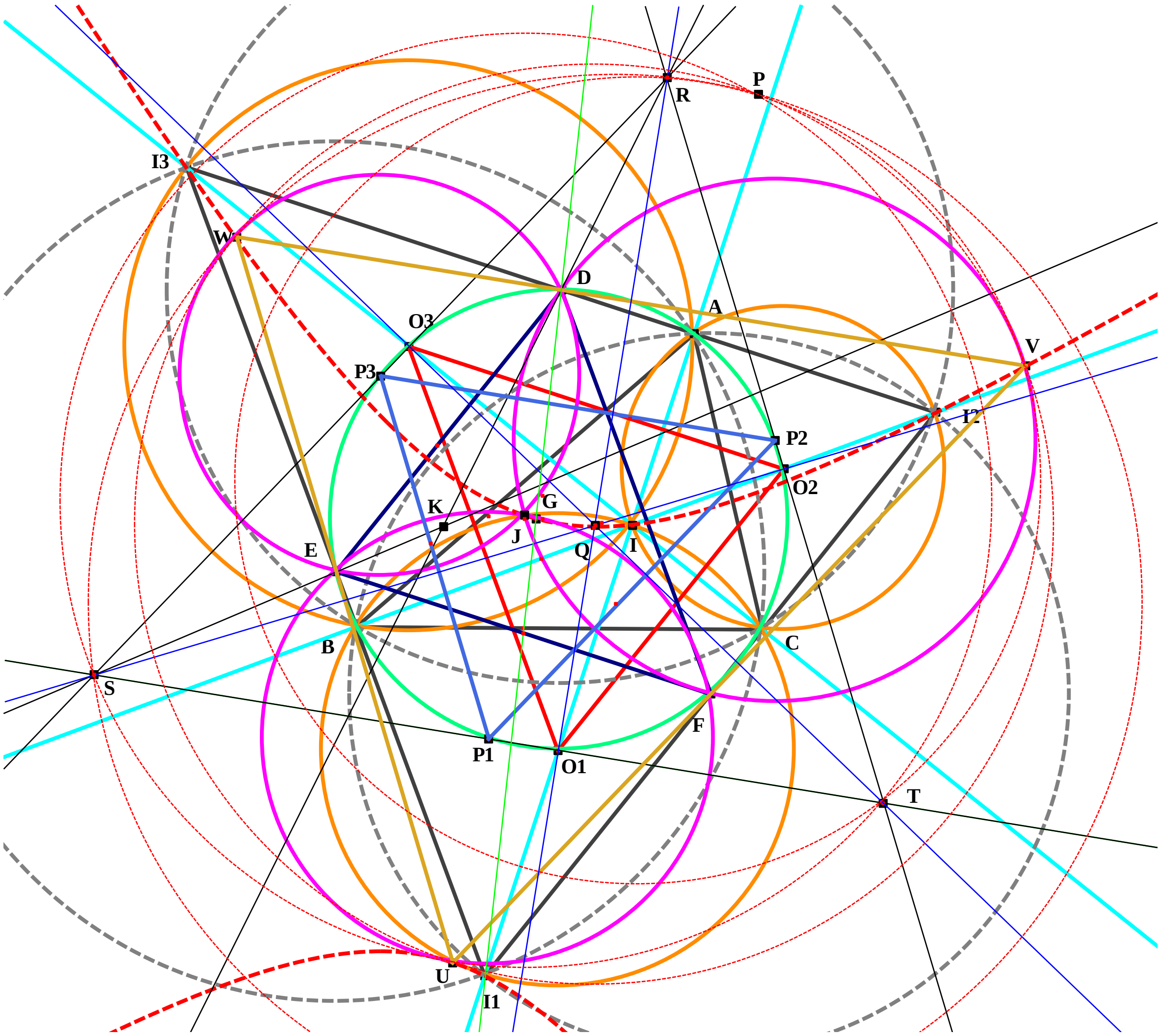

Fig. 1
The two sets of excentres I_1, I_2, I_3 and U, V, W

1. Introduction

The best way of proceeding is to describe the various stages whereby Fig. 1 is constructed mentioning properties that are exhibited as they emerge.

We start with a triangle ABC, its incentre I and its triangle of excentres $I_1I_2I_3$. The segments I_2I_3, I_3I_1, I_1I_2 have midpoints D, E, F, the circle ABCDEF being the nine-point circle of triangle $I_1I_2I_3$. Thus far we have the same configuration of Article 96 (CJB/2010/96). J is the incentre of triangle DEF and G its centroid. The deLongchamps point of DEF is proved in Article 96 to coincide with I the incentre of ABC. A nine-point rectangular hyperbola passes through points I, J, I_1, I_2, I_3, U, V, W, G, as described in Article 95 (CJB/2010/95). What now follows appears only in this article.

Circles I_1BC, I_2CA, I_3AB are now drawn and their centres O_1, O_2, O_3 are shown to lie on the circumcircle of ABC. These circles all pass through I. Interestingly the lines I_2I_3, EF and O_2O_3 are parallel as are similar sets of lines including FD and DE. In fact the figures EFO_2O_3, FDO_3O_1 and DEO_1O_2 are rectangles.

Circles UBC, VCA, WAB are now drawn and their centres P_1, P_2, P_3 are shown to lie on the circumcircle of ABC. These circles all pass through J. The well known circles I_1I_2AB, I_2I_3BC, I_3I_1CA centres F, D, E also appear in Fig. 1.

From this point on the results mentioned are not proved, but are indicated by *CABRI II plus.* Lines O_1P_1, O_2P_2, O_3P_3 form a triangle RST. The midpoints of the sides of R, S, T are P_1, P_2, P_3. The points R, S, T are in fact the excentres of triangle $O_1O_2O_3$. The two triangles $O_1O_2O_3$ and $P_1P_2P_3$ with vertices all lying on circle ABCDEF now take on the same properties as ABC and DEF, but with excentres R, S, T rather than I_1, I_2, I_3 This process could apparently be carried on *ad infinitum.*

The lines RI_1, SI_2, TI_3 are concurrent at Q, the incentre of triangle $O_1O_2O_3$. It is also indicated that circles RSW, STU, TRV, UVW all pass through a point P.

2. The circles I_1BC, I_2CA, I_3AB

The co-ordinates of the excentres are $I_1(-a, b, c)$, $I_2(a, -b, c)$, $I_3(a, b, -c)$. It is now straightforward to find the equations of the circles I_1BC, I_2CA, I_3AB, which are

I_1BC: $$bcx^2 - a^2yz + b(c - b)zx + c(b - c)xy = 0, \quad (2.1)$$
I_2CA: $$cay^2 - b^2zx + c(a - c)xy + a(c - a)yz = 0, \quad (2.2)$$
I_3AB: $$abz^2 - c^2xy + a(b - a)yz + b(a - b)zx = 0. \quad (2.3)$$

Now the centre of a conic with equation
$$ux^2 + vy^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \quad (2.4)$$
is $(vw - gv - hw - f^2 + fg + hf,\ wu - hw - fu - g^2 + gh + fg,\ uv - fu - gv - h^2 + hf + gh)$ see Bradley [1], so from Equations (2.1) to (2.3) we may obtain the co-ordinates of the centres O_1, O_2, O_3 of the three circles. They are
$$O_1(-a^2, b(b + c), c(b + c)),\ O_2(a(c + a), -b^2, c(c + a)),\ O_3\quad (a(a + b), b(a + b), -c^2) \quad (2.5)$$
It may now be checked that O_1, O_2, O_3 all lie on the circumcircle of ABC with equation
$$a^2yz + b^2zx + c^2xy = 0. \quad (2.6)$$

It may also be checked that all three circles pass through I(a, b, c), the incentre of triangle ABC.

3. The points D, E, F and some of their properties

The points D, E, F are the midpoints of I_2I_3, I_3I_1, I_1I_2 respectively so their co-ordinates are

$$D(a^2, bc - b^2, bc - c^2), E(ca - a^2, b^2, ca - c^2), F(ab - a^2, ab - b^2, c^2). \quad 3.1)$$

It may be checked that D, E. F lie on the circumcircle of ABC. (In fact the circle ABCDEF is the nine-point circle of triangle $I_1I_2I_3$, D, E, F being the midpoints of the sides.) The points U, V, W are the excentres of triangle DEF and it follows from Section 2 that the circles UEF, VFD, WDE all pass through J, the incentre of triangle DEF and that the centres of the three circles P_1, P_2, P_3 lie on the circumcircle of triangle DEF, which is the same as circle ABC.

The co-ordinates of U, V, W, J, P_1, P_2, P_3 are very complicated and we do not record them. The well known circles I_1I_2AB, I_2I_3BC, I_3I_1CA centres F, D, E also appear in Fig. 1. The circle BCI_1I_2 has equation

$$bcx^2 + a^2yz + b(b + c)zx + c(b + c)xy = 0. \quad (3.2)$$

The equations of CAI_3I_1, ABI_1I_2 may be written down from Equation (3.2) by cyclic change of x, y, z and a, b, c.

4. Further properties of the key points

The equations of O_2O_3, I_2I_3 are respectively

$$bcx - c(a + c)y - b(a + b)z = 0 \quad (4.1)$$

and

$$cy + bz = 0. \quad (4.2)$$

and these are parallel since they meet at (c – b, b, – c) a point on the line at infinity x + y + z = 0.

The equation of EF is

$$bcx + c(a - c)y + b(a - b)z = 0 \quad (4.3)$$

and this too passes through (c – b, b, – c) and hence is parallel to both O_2O_3 and I_2I_3. Similarly O_3O_1, I_3I_1, FD are parallel and O_1O_2, I_1I_2, DE are parallel.

The displacement O_3E is (a(b + c), – ab, – ac)/{(a + b + c)(a – b – c)} and the displacement EF is (a(b – c), ab, ac)/{(a + b – c)(a – b + c)}.

Now if (f, g, h) and (u, v, w) are two displacements (with f + g + h = 0 and u + v + w = 0, of course) then these displacements are at right angles if, and only if,

$$a^2(hv + gw) + b^2(fw + hu) + c^2(gu + fv) = 0. \quad (4.4)$$

It may now be checked that O_3E and EF are at right angles. It follows that O_3EFO_2 is a rectangle as are O_1FDO_3 and O_2DEO_1.

Reference

1. C. J. Bradley, *The Algebra of Geometry*, Highperception, Bath, UK (2007).

Flat 4,
Terrill Court,
12-14, Apsley Road,
BRISTOL BS8 2SP.