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A Nine Point Rectangular Hyperbola

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Abstract: The configuration consisting of the rectangular hyperbola passing through the incentre, the excentres, the centroid and deLongchamps point in a triangle and three other significant points exhibits some interesting properties which are investigated.

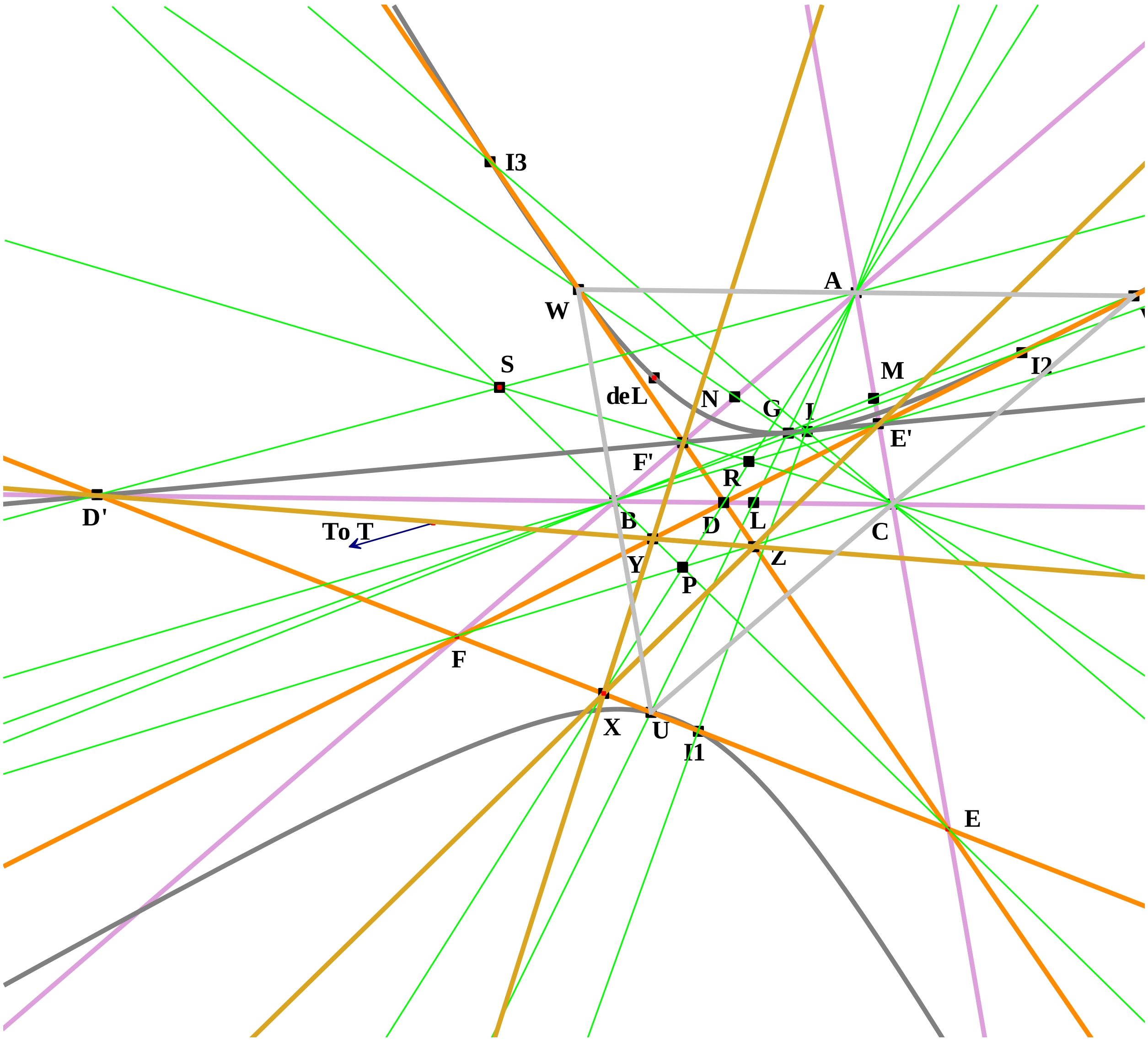

Fig. 1
The nine point rectangular hyperbola

1. Introduction

Given a triangle ABC let the midpoints of the sides BC, CA, AB be denoted by L, M, N respectively. Points U, V, W on the extensions of AL, BM, CN respectively are defined so that AU = 2AL, BV = 2BM, CW = 2CN. The incentre I and the three excentres I_1, I_2, I_3, the centroid G and deLongchamps point deL and the points U, V, W are the nine points we consider and first we prove that they lie on a conic, which is a rectangular hyperbola, since the incentre and the three excentres form an orthocentric quartet.

The three lines UI_1, VI_2, WI_3 form a triangle with vertices D, E, F which we show lie on BC, CA. AB respectively and it is proved that AD, BE, CF are concurrent at a point P. Points D', E', F' are the harmonic conjugates of D, E, F on BC, CA, AB respectively. It is a consequence that D', E', F' are collinear. It is also shown that lines AD, BE', CF' are concurrent at a point R, AD', BE, CF' are concurrent at a point S and AD', BE', CF are concurrent at a point T. If AD meets EF at X, then X is the harmonic conjugate of D' with respect to E and F and similarly for Y and Z.

Areal co-ordinates are used throughout with ABC as triangle of reference.

2. The rectangular hyperbola through U,V,W, I, I_1, I_2, I_3, G, deL

The conic through the five points I, I_1, I_2, I_3, G is easily obtained from their co-ordinates I(a, b, c), I_1(– a, b, c), I_2(a, – b, c), I_3(a, b, – c), G(1, 1, 1) and is

$$(b^2 - c^2)x^2 + (c^2 - a^2)y^2 + (a^2 - b^2)z^2 = 0. \qquad (2.1)$$

This conic clearly passes through U(– 1, 1, 1), V(1, – 1, 1), W(1, 1, – 1). Rather amazingly it passes through deLongchamps point deL whose x-co-ordinates is $-3a^4 + 2a^2(b^2 + c^2) + (b^2 - c^2)^2$ and whose y- and z- co-ordinates follow by cyclic change of a, b, c. The conic is a rectangular hyperbola, since the incentre and the three excentres form an orthocentric quartet.

3. The points D, E, F and their properties

The point D is the intersection of lines VI_2 and WI_3. The equations of these lines are

$$(b - c)x + (a - c)y + (a - b)z = 0, \qquad (3.1)$$

and

$$(b - c)x + (c - a)y + (b - a)z = 0. \qquad (3.2)$$

The point D therefore has co-ordinates D(0, a – b, c – a). Similarly E has co-ordinates E(a – b, 0, b – c) and F has co-ordinates (c – a, b – c, 0). These points lie on the sides BC, CA, AB respectively and evidently AD, BE, CF are concurrent at P with co-ordinates

$$P(1/(b - c), 1/(c - a), 1/(a - b)). \qquad (3.3)$$

4. The points D', E', F', R, S, T, X, Y, Z

The point D' is the harmonic conjugate of D with respect to B and C and so has co-ordinates D'(0, b – a, c – a). Similarly E' and F' have co-ordinates E'(a – b, 0, c – b) and F'(a – c, b – c, 0). Since D, E, F are the feet of Cevians, the points D', E', F' are collinear.

The point R is the point of concurrence of the three lines AD, BE', CF' and so has co-ordinates R(1/(c – b), 1/(c – a), 1/(a – b)). Similarly S and T have co-ordinates S(1/(b – c), 1/(a – c), 1/(a – b)), T(1/(b – c), 1/(c – a), 1/(b – a)).

The point X is the intersection of AD with EF and since ABF and ACE are straight lines and the range {B, C; D, D'} is harmonic then it follows that D' and X separate E and F harmonically. Similarly for Y and Z.

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