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Circles formed by an Isosceles Trapezium

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Abstract Given an isosceles trapezium ABCD with AD parallel to BC, if tangents to the cyclic quadrilateral ABCD are drawn to produce six points of intersection, then six more circles may be drawn, all passing through the centre O of ABCD. Five of these circles are obvious but the fact that the sixth circle passes through O is an interesting result and in this paper a proof is given using Cartesian co-ordinates.

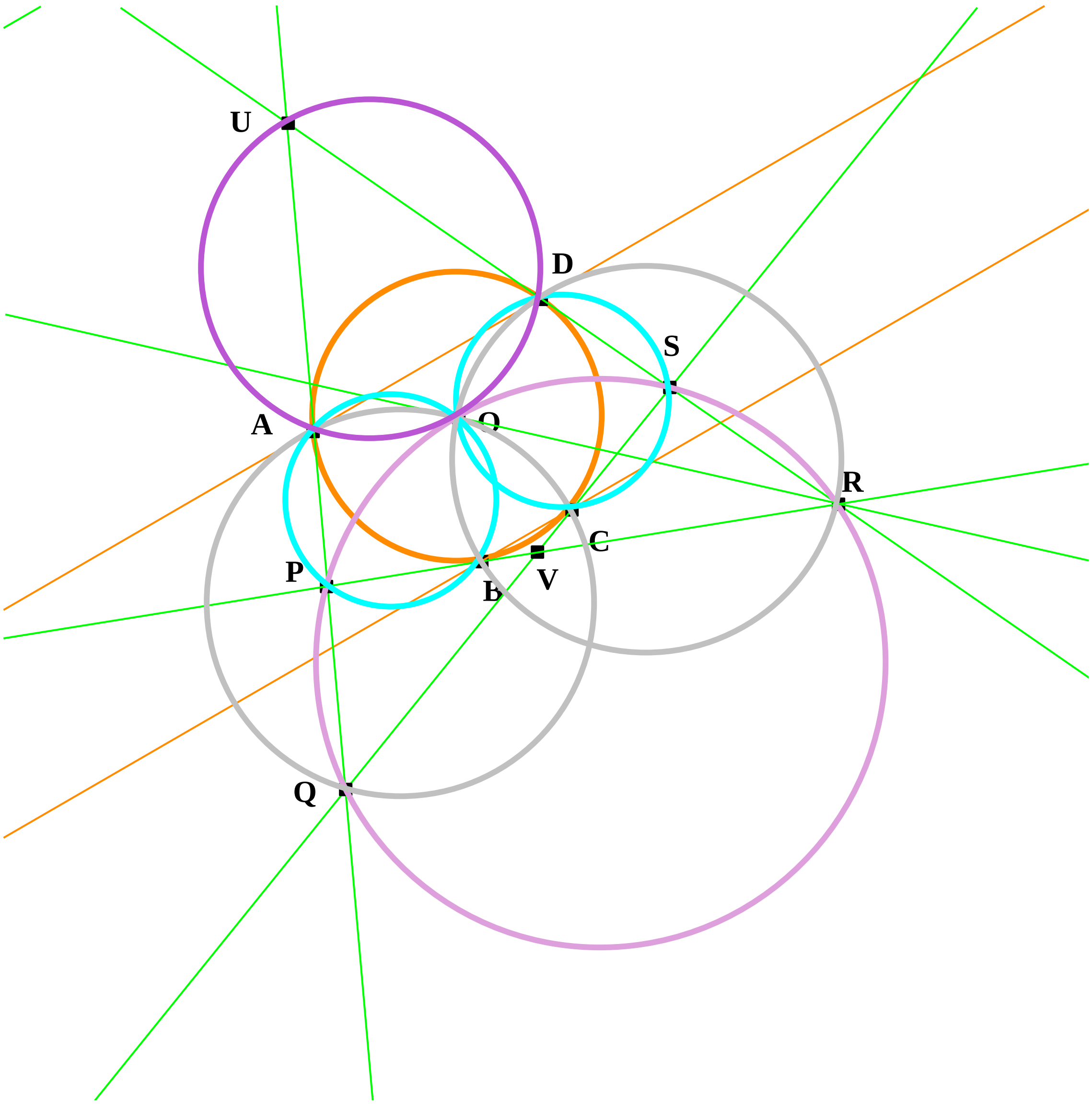

Fig. 1
The Seven Circles

1. Introduction

In Fig.1 ABCD is an isosceles trapezium with AD parallel to BC. The centre of the circle ABCD is labelled O and P, Q, R, S are defined as the intersections of that tangents at A, B and A, C and B, D and C, D respectively. U, V are defined as the intersections of the tangents at A, D and B, C respectively. It is immediate from the existence of right angles between radii and tangents that the following circles exist: (i) DUAO, (ii) AQCO, (iii) DRBO, (iv) APBO, (v) DSCO. Fig. 1 indicates that the (obvious) circle PQRS also passes through O. In this brief article we prove this by using Cartesian co-ordinates with ABCD as unit circle.

2. Tangents and Points

We take A to have co-ordinates $((1 - a^2)/(1 + a^2), 2a/(1 + a^2))$ and B, C, D to have similar co-ordinates but with parameters b, c, d respectively.

Since AD is parallel to BC we have

$$(1 - ad)/(a + d) = (1 - bc)/(b + c), \qquad (2.1)$$

which, on simplification yields the condition

$$abc + bcd - abd - acd - a + b + c - d = 0. \qquad (2.2)$$

The equation of the tangent at A is

$$(1 - a^2)x + 2ay = (1 + a^2), \qquad (2.3)$$

The tangents at B, C, D have similar equations but with parameters b, c, d respectively.

P is the intersection of the tangents at A and B and so has co-ordinates

$$((1 - ab)/(1 + ab), (a + b)/(1 + ab)). \qquad (2.4)$$

Similarly Q, R, S have co-ordinates with pairs of parameters (a, c), (b, d), (c, d).

We aim to show that circle PQRS passes through O.

3. Circle PQRS

We suppose that circle PQRS has an equation of the form

$$x^2 + y^2 + 2gx + 2fy + k = 0 \qquad (3.1)$$

and the circle passes through O if, and only if $k = 0$.

We insert into Equation (3.1) the co-ordinates of P, Q, R and get three equations for g, f and k.

The value of k obtained is

$$(1 + b^2)(a^3(bc - bd - cd - 1) \backslash + a^2(bcd + b + c - d) + a(bc - bd - cd - 1) + bcd + b + c - d) \qquad (3.2)$$

and by virtue of Equation (2.2) this vanishes.

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