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Perpendiculars in a Cyclic Quadrilateral

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Fig. 1

Perpendiculars drawn in ABCD produce four more Cyclic Quadrilaterals

Abstract: Perpendiculars from A and C on to opposite sides of a cyclic quadrilateral produce four more cyclic quadrilaterals and two sets of parallel lines one set containing five lines and the other set containing six lines.

1. Introduction

To be clear about the configuration in Fig.1 it needs to be described in the order in which the various lines and circles are drawn. First drawn is a general cyclic quadrilateral ABCD. Then perpendiculars from A are drawn onto BC and CD meeting them at P and Q respectively. Similarly perpendiculars from C on to AB and DA are drawn meeting them at R and S respectively. It is the found that PQRS is a circle centre X passing through A and C. Point U is the intersection of AP and CR and point W is the intersection of AQ and CS. It is now found that AUCW is cyclic with circle centre Y on OX produced, where OX = XY. Also BPUR and DQWS are cyclic quadrilaterals whose circles are of equal radius. Also cyclic quadrilaterals AUCW and ABCD have equal angles (with angle B = angle W etc.).

External diagonal points are now found. AB and DC meet at F and AD and BC meet at G. AU and WC meet at H and AW and UC meet at K. Further points in the figure are (i) J is the intersection of AB and WC, (ii) M is the intersection of AD and UC, (iii) L is the intersection of AU and DC, (iv) N is the intersection of AW and BC. The construction ensures that the following are straight lines: ARBJ, AUPHL, ADSGM, AQWK, RUCMK, and BPCG. Remarkably from the points defined the following lines are parallel: BD, JL, MN, PS, RQ and UW and the following lines also parallel: OXY, BU, DW, HG and FK. It also transpires that JL = MN.

In the following sections we prove these results using Cartesian co-ordinates with circle ABCD the unit circle, though the right angles are used in dealing with three of the cyclic quadrilaterals.

2. ABCD and the points P, Q, R, S and the circle PQRS

We take A to have co-ordinates (0, 1) and B, C, D to have parameters b, c, d so that B, for example, has co-ordinates $((1 - b^2)/(1 + b^2), 2b/(1 + b^2))$. The equation of the line AB is

$$(1 - b)x + (1 + b)y = 1 + b. \qquad (2.1)$$

The equation of the line perpendicular to AB through C therefore has equation

$$(1 + c^2)((1 + b)x - (1 - b)y) = (1 - c^2)(1 + b) - 2c(1 - b). \qquad (2.2)$$

The point R where these two lines meet has co-ordinates (x, y), where

$$x = \{(1 - c)(b^2c + b(1 + c) + 1)\}/\{(1 + b^2)(1 + c^2)\},$$
$$y = \{b^2(1 + c) + b(1 - c)^2 + c(1 + c)\}/\{(1 + b^2)(1 + c^2)\}. \qquad (2.3)$$

Similarly the point S has co-ordinates (x, y), where

$$x = -\{(1 + d)(c^2d + c(1 - d) - 1)\}/\{(1 + c^2)(1 + d^2)\},$$
$$y = \{c^2(1 + d) + c(1 - d)^2 + d(1 + d)\}/\{(1 + c^2)(1 + d^2)\}. \qquad (2.4)$$

And the point P has co-ordinates (x, y), where

$$x = \{(1 - c)(b^2c - b(1 + c) + 1)\}/\{(1 + b^2)(1 + c^2)\},$$
$$y = \{b^2c(1 + c) + b(1 - c)^2 + 1 + c)\}/\{(1 + b^2)(1 + c^2)\}. \quad (2.5)$$

And finally the point Q has co-ordinates (x, y), where

$$x = \{(1 - d)(c^2d - c(1 + d) + 1)\}/\{(1 + c^2)(1 + d^2)\},$$
$$y = \{c^2d(1 + d) + c(1 - d)^2 + d + 1\}/\{(1 + c^2)(1 + d^2)\}. \quad (2.6)$$

It is, of course, immediate from the diagram, because of the right angles that the midpoint X of AC is the centre of the circle PQRS passing through A and C. This also follows by calculation and the equation of circle PQRS is

$$(x^2 + y^2)(1 + c^2) - (1 - c^2)x - (1 + c)^2y + 2c = 0. \quad (2.7)$$

The co-ordinates of X, the centre of circle PQRS are $X((1 - c^2)/\{2(1 + c^2)\}, (1 + c)^2/\{2(1 + c^2)\}$.

(2.8)

The radius of the circle PQRS is $|1 - c|/\sqrt{(2(1 + c^2))}$.

3. Circle AUCW

Simple angle chasing shows that AUCW is a cyclic quadrilateral, and also that its angles are the same as those of ABCD (with angle B = angle W and angle D = angle U). However we wish to obtain its equation.

U is the intersection of AP and RC whose equations are respectively

$$(b + c)x + (bc - 1)(y - 1) = 0, \quad (3.1)$$

and

$$(b + 1)(1 + c^2)x + (b - 1)(1 + c^2) + b(c^2 - 2c - 1) + c^2 + 2cv - 1 = 0.$$

(3.2)

So U has co-ordinates (x, y), where

$$x = 2(1 - b^2c^2)/\{(1 + b^2)(1 + c^2)\}, \quad (3.3)$$
$$y = \{(1 + b^2)(1 + c)^2 + 2b(1 + c^2)\}/\{(1 + b^2)(1 + c^2)\}. \quad (3.4)$$

Similarly W has co-ordinates (x, y), where

$$x = 2(1 - c^2d^2)/\{(1 + c^2)(1 + d^2)\}, \quad (3.5)$$
$$y = \{(1 + c^2)(1 + d)^2 + 2c(1 + d^2)\}/\{(1 + c^2)(1 + d^2)\}. \quad (3.6)$$

The equation of circle AUCW may now be obtained and is

$$(1 + c^2)(x^2 + y^2) - 2(1 - c^2)x - 2(1 + c)^2y + c^2 + 4c + 1 = 0. \quad (3.7)$$

The centre Y of circle AUCW is Y, with co-ordinates $((1 – c^2)/(1 + c^2), (1 + c)^2/(1 + c^2))$. It follows that O, X, Y are collinear and OX = XY. The radius of circle AUCW is 1, the same as circle ABCD. However the two cyclic quadrilaterals are not similar.

The gradient of OXY is $(1 + c)/(1 – c)$ and both DW and BU have the same slope and are all therefore parallel. Simple angle chasing shows that BPUR and DQWS are equiangular and though BU = DW, these cyclic quadrilaterals are not congruent.

4. The points F, G, K, H, L, M, J, N

These eight points are defined as follows F = AB^CD, G = AD^BC (the exterior diagonal points of ABCD), K = AW^UC, H = AU^CD (the exterior diagonal points of AUCW), L = AU^CD, M = AD^UC, J = AB^WC and N = BC^AW.

We give the co-ordinates of these eight points:

F: $((1 – d)(b + 1)(c – 1), 2(cd – b))/(bcd – bc – bd + cd – b + c + d – 1)$; (4.1)

G: $((1 – c)(b – 1)(d + 1), 2(bc – d))/bcd + bc – bd – cd + b + c – d – 1)$: (4.2)

K: $((2(bc + 1)(c^2d – cd – c + 1), (b(c^3(d + 1) + c^2(d – 3) – c(d + 1) – d – 1) + c^3(d + 1) + c^2(d + 1) + c(3d – 1) – d – 1))/((1 + c^2)(b(c(d – 1) – d – 1) + c(d + 1) + d – 1)$; (4.3)

H: $((2(1 – bc)(c^2d + c – cd – 1), (b(c^3(d + 1) + c^2(d + 1) + c(3 – d) – d – 1) + c^3(d + 1) + c^2(1 – 3d) – c(d + 1) – d – 1))/((1 + c^2)(b(c(d + 1) – d + 1) + c(1 – d) – d – 1)$; (4.4)

L: $((1 – d)(c – 1)(bc – 1), b(c^2d + c(d – 1) + 1) + c^2d + c(1 – d) + 1)/(1 + c^2)(bd + 1)$; (4.5)

M: $(((1 – c)(d + 1)(bc + 1), b(c^2 + c(d – 1) + d) + c^2 + c(1 – d) + d)/(1 + c^2)(bd + 1)$; (4.6)

J: $(– (1 + b)(c^2d + c(1 – d) – 1), b(c(d – 1) + d + 1) + c(c(d + 1) – d + 1)/(1 + c^2)(bd + 1)$;(4.7)

N: $(1 – b)(c^2d – c(d + 1) + 1), bc(c(d + 1) + d – 1) + c(1 – d) + d + 1)/(1 + c^2)(bd + 1)$; (4.8)

5. The two sets of parallel lines

The gradients of the various lines may now be calculated and it is found that the following five lines OXY, BU, DW, HG, FK have gradients $(1 + c)/(1 – c)$, so these lines are parallel to each other all being perpendicular to AC. These gradients are independent of the positions of B and D.

It is also found that the following lines BD, JL, MN, PS, RQ, UW have gradients$(bd – 1)/(b + d)$ and are therefore parallel to one another in the direction of BD. Interestingly MN and JL are not only parallel but are equal displacements.

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