Article: CJB/2010/86

The Perpendiculars to Three Segments at a Point determine a Conic

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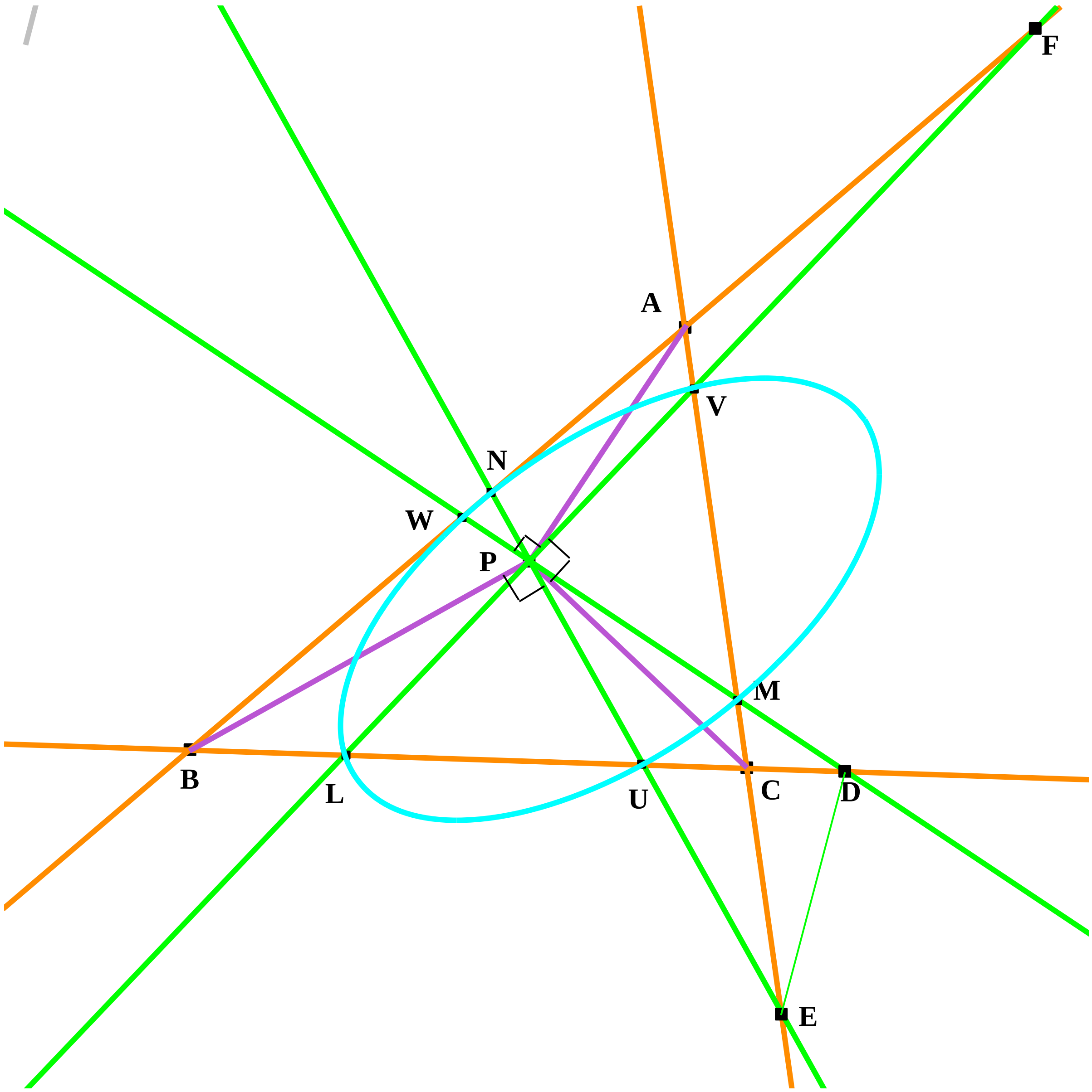

Fig. 1
Perpendiculars to AP, BP, CP determine a Conic LMNUVW

Abstract: In the plane of a triangle ABC a general point P is chosen. Lines perpendicular to AP and BP and CP meet the sides at W, M and U, N and V, L respectively. We prove that the points L, M, N, U, V, W always lie on a conic and that the third points of intersection D, E, F lie on a line.

1. Introduction

The important result is that if the perpendiculars to AP, BP, CP through P meet sides BC, CA, AB respectively at D, E, F then DEF is a straight line. It then follows immediately from the result of Article : CJB/2010/85 [1], since DM passes through W on AB, EN passes through U on BC and FL passes through V on CA, that points L, M, N, U, V, W lie on a conic. We prove that DEF is a straight line, using areal co-ordinates with ABC as triangle of reference.

2. The displacements AP and WM

Suppose P has co-ordinates (l, m, n), where l + m + n = 1. Then the displacement **AP** = (l – 1, m, n). Let the line WPM have equation ux + vy + wz = 0. We have to write down the conditions so that (i) P lies on the line and (ii) the displacement **WM** is perpendicular to **AP.**

The first is easy enough and is

$$ul + vm + wn = 0. \qquad (2.1)$$

Suppose now M has co-ordinates (p, q, r) the p = – w/(u – w), q = 0 and r= u/(u – w). Similarly W has co-ordinates (– v/(u – v), u/(u – v), 0) and the displacement **WM** is (x, y, z), where

$$x = u(v - w)/\{(u - v)(u - w)\}, \qquad y = u/(v - u), \qquad z = u/(u - w) \qquad (2.2)$$

Writing f = l – 1, g = m, h = n condition (ii) that **WM** is perpendicular to **AP** is

$$a^2(gz + hy) + b^2(hx + fz) + c^2(fy + gx) = 0. \qquad (2.3)$$

Values of (u, v, w) satisfying equations (2.1) and (2.3) are

$$u = -2a^2mn - (l - m - n - 1)(b^2n + c^2m), \qquad (2.4)$$

$$v = a^2n(l - m + n) + b^2n(1 - 2l) + c^2(l^2 - l(m - n + 1) - n), \qquad (2.5)$$

$$w = a^2m(l + m - n) + b^2(l^2 + l(m - n + 1) - m) + c^2m(1 - 2l). \qquad (2.6)$$

Unnormalised co-ordinates of M and N now follow and those of M are (x, y, z) where

$$x = a^2m(l + m - n) + b^2(l^2 + l(m - n - 1) - m) + c^2m(1 - 2l). \qquad (2.7)$$

$$y = 0, \qquad (2.8)$$

$$z = 2a^2mn + (l - m - n - 1)(b^2n + c^2m). \qquad (2.9)$$

And those of W are (x, y, z), where

$$x = a^2n(l - m + n) + b^2n(1 - 2l) + c^2(l^2 - l(m - n + 1) - n), \qquad (2.10)$$

$$y = 2a^2mn + (l - m - n - 1)(b^2n + c^2m), \qquad (2.11)$$

$$z = 0. \qquad (2.12)$$

3. The co-ordinates of D, E, F

The equation of the line WM may now be written down and it is found that it meets BC at the point D with co-ordinates (x, y, z), where

$$x = 0, \tag{3.1}$$
$$y = -(a^2m(l + m - n) + b^2(l^2 + l(m - n - 1) - m) + c^2m(1 - 2l)), \tag{3.2}$$
$$z = a^2n(l - m + n) + b^2n(1 - 2l) + c^2(l^2 - l(m - n + 1) - n). \tag{3.3}$$

The co-ordinates of E, F follow by cyclic change of x, y, z and a, b, c and l, m, n.

4. DEF is a straight line

The ratio BD/DC follows from equations (3.2) and (3.3) and similarly CE/EA and AF/FB may be obtained. It then follows, since $l + m + n = 1$, that $(BD/DC)(CE/EA)(AF/FB) = -1$ and hence by Menelaus' theorem that DEF is a straight line. That a conic passes through L, M, N, U, V, W follows at once from the result of Article 85.

Reference

1. C. J. Bradley, Article: CJB/2010/85 of this series.

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