Article 79

Conics in the Ex-circle Configuration

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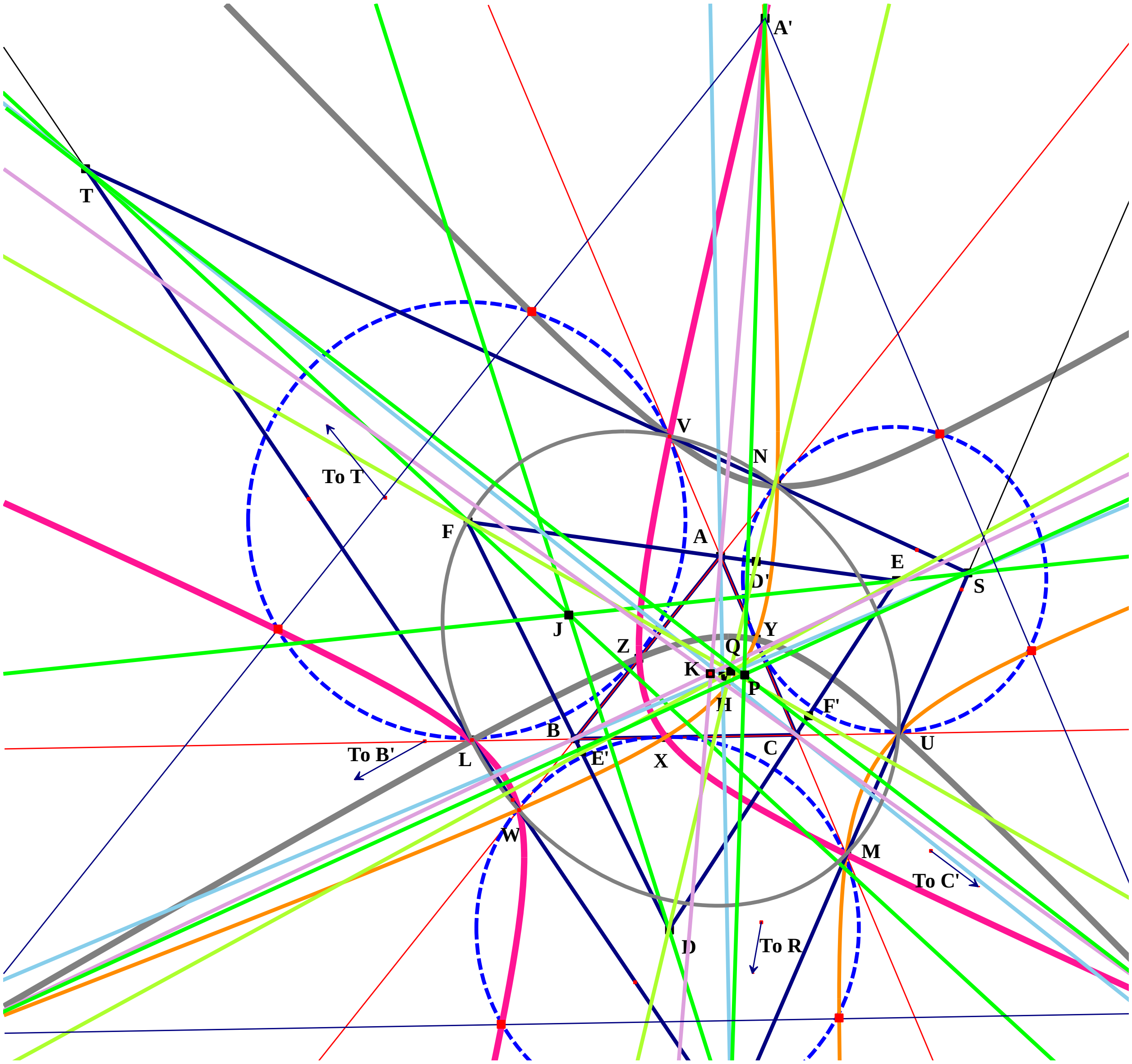

1. Introduction

Given a triangle ABC there are three ex-circles. The one opposite A has centre D and touches BC at X, CA at M and AB at W. The one opposite B has centre E and touches CA at Y, AB at N and BC at U. The one opposite C has centre F and touches AB at Z, BC at L and CA at V.

In what follows we show that a conic passes through L, M, N, U, V, W and that three further conics exist passing through (i) L, U, Z, Y, V, N, (ii) M, V, X, Z, W, L and (iii) N, W, Y, X, U, M. These conics involve the points of contact with the sides of pairs of ex-circles.

Lines LW, MU meet at R; MU, NV meet at S; NV, LW meet at T. It follows that lines SE, TF and RD are concurrent at a point J. It also follows that lines RA, SB and TC are concurrent at H, the orthocentre of triangle ABC.

It is then proved that the centre of conic LUZYVN is a point D' lying on EF, the centre of conic MVXZWL is a point E' lying on FD and the centre of conic NWYXUM is a point F' lying on DE. Furthermore DD', EE', FF' are concurrent at a point Q.

Conics MVXZWL and NWYXUM obviously meet at M, X and W but their fourth point of intersection determines a point A', with B', C' similarly defined. It is proved that AA', BB', CC' meet at the symmedian point K of triangle ABC and also triangle A'B'C' is an enlargement of triangle ABC. Finally it is shown that the lines RA', SB', TC' are concurrent at a point P.

Areal co-ordinates are used with ABC the triangle of reference. *DERIVE* and *CABRI II plus* were used in the drawing and analysis.

2. Revision of the co-ordinates of main points and the equations of the ex-circles

The centres of the ex-circles are D(– a, b, c), E(a, – b, c), F(a, b, – c). Points of contact of the ex-circles with interior points of the sides of ABC are X(0, c + a – b, a + b – c), Y(b + c – a, 0, a + b – c), Z(b + c – a, c + a – b, 0). The other six points of contact of the ex-circles with exterior points of the sides of ABC are U(0, a – b – c, a + b + c), V(a + b + c, 0, b – c – a), W(c – a – b, a + b + c, 0), L(0, a + b + c, a – b – c), M(b – c – a, 0, a + b + c), N(a + b + c, c – a – b, 0).

The ex-circle opposite A passing through W, X and M is

$$(a + b + c)^2x^2 + (a + b - c)^2y^2 + (a - b + c)^2z^2 - 2(a^2 - (b - c)^2)yz + 2((c + a)^2 - b^2)zx + 2((a + b)^2 - c^2)xy = 0. \qquad (2.1)$$

The ex-circles opposite B and C may be written down by cyclic change of x, y, z and a, b, c.

3. The four conics

The equation of the conic LMNUVW through the external points of contact is

$$(a^2 - (b + c)^2)(a^2 - (b - c)^2)x^2 + (b^2 - (c + a)^2)(b^2 - (c - a)^2) y^2 + (c^2 - (a + b)^2)(c^2 - (a - b)^2)z^2 - 2(a^2 + (b + c)^2) (a^2 - (b - c)^2)yz - 2(b^2 + (c + a)^2)(b^2 - (c - a)^2)zx - 2(c^2 + (a + b)^2)(c^2 - (a - b)^2)xy = 0. \qquad (3.1)$$

The equation of the conic LUZYVN through the points of contact of the ex-circles opposite B and C is

$$(a^2 – (b – c)^2)x^2 + (a^2 – (b + c)^2)y^2 + (a^2 – (b + c)^2)z^2 – 2(a^2 + (b + c)^2)yz + 2(a^2 + b^2 – c^2)zx + 2(a^2 – b^2 + c^2)xy = 0. \qquad (3.2)$$

The equations of the conics MVXZWL and NWYXUM may be written down from equation (3.2) by repeated cyclic change of x, y, z and a, b, c.

4. Points R, S, T and the concurrence of RD, SE, TF

The equation of UM is

$$(a^2 – (b + c)^2x – ((a + c)^2 – b^2)y + ((a – b)^2 – c^2)z = 0. \qquad (4.1)$$

The equations of VN and WL may now be written down by repeated cyclic change of x, y, z and a, b, c.

UM and VN meet at the point S whose co-ordinates are (x, y, z), where

$$x = ((a + c)^2 – b^2)(a^2 + b^2 – c^2),$$
$$y = 2b(c + a)((c – a)^2 – b^2), \qquad (4.2)$$
$$z = ((c + a)^2 – b^2)(b^2 + c^2 – a^2).$$

The equation of SE is

$$b(a^4 + 2a^2(c^2 – b^2) + (b^2 – c^2)(b^2 + 3c^2))x +((c + a)^2 – b^2)(a^3 + a^2c – a(b^2 + c^2)+ c(b^2 – c^2))y + b(3a^4 – 2a^2(b^2 + c^2) – (c^2 – b^2)^2)z = 0. \qquad (4.3)$$

The equations of TF and RD may now be written down by repeated cyclic changes of x, y, z and a, b, c.

The determinant whose rows are the coefficients of x, y, z in the equations of SE, TF, RD may now be shown to vanish, showing these lines are concurrent at a point J. The actual co-ordinates of J are technically very complicated and lengthy and are therefore not recorded here.

5. RA, SB, TC meet at H

The equation of the line RA is $(c^2 + a^2 – b^2)y = (a^2 + b^2 – c^2)z$ and RA clearly passes through H, the orthocentre of ABC, as do SB and TC.

6. The centres of the conics

The centre D' of the conic LUZYVN has co-ordinates $(-a^2, b(c-b), c(b-c))$ and the centres E', F' of the conics MVXZWL and NWYXUM may be written down by repeated cyclic changes of x, y, z and a, b, c. It may now be shown that DD', EE', FF' are concurrent at a point Q with co-ordinates $(a(3a-b-c), b(3b-c-a), c(3c-a-b))$.

7. Where the conics meet

The conics MVXZWL and NWYXUM meet at M, X, W and at a fourth point A' whose co-ordinates are $(-(b+c)(ab+ca+b^2+c^2), b^2(a+b+c), c^2(a+b+c))$. Points B', C' may be similarly defined and clearly AA' passes through the symmedian point K of triangle ABC. The following facts are also easily checked: triangle A'B'C' is an enlargement of triangle ABC centre K, which is also the symmedian point of triangle A'B'C'.

8. Lines RA', SB', TC' are concurrent

The equation of the line RA' is

$$(b-c)(a^2-(b+c)^2)x + (a^2(b+c) + 2ca(c-b) - (b+c)(b^2-c^2))y - (a^2(b+c) + 2ab(b-c) + (b+c)(b^2-c^2))z = 0. \qquad (8.1)$$

The lines SB', TC' have equations that may be written down from equation (8.1) by repeated cyclic change of x, y, z and a, b, c.

The determinantal method involving the coefficients of x, y, z in these equations show that RA', SB', TC' are concurrent at a point P, which has lengthy and technically involved expressions.

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