# Article 72

## What happens when you reflect a Triangle in any Line

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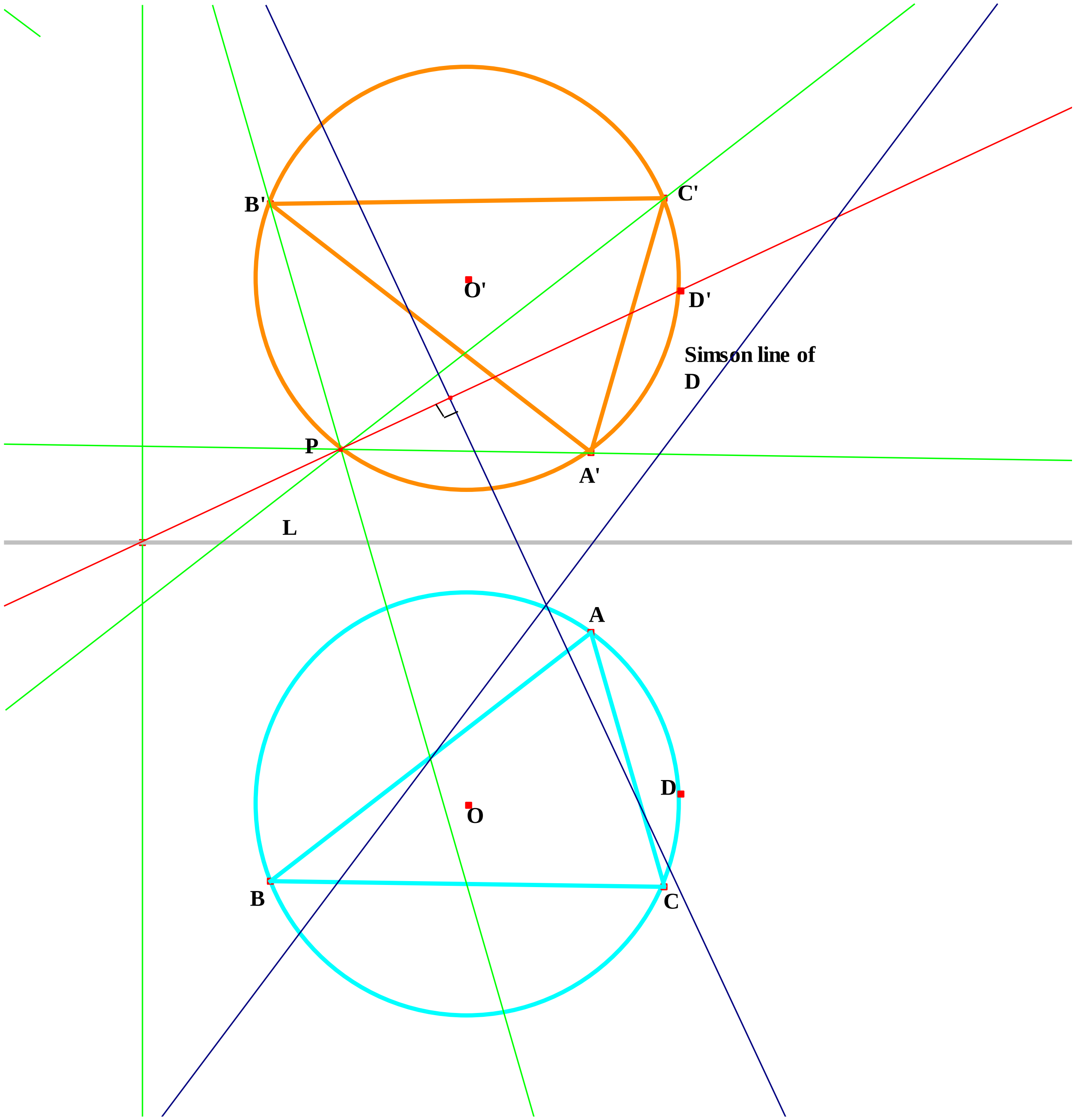


### 1. Introduction

In the figure a triangle ABC and its circumcircle are shown together with a reflection line L and the image triangle A'B'C'. What happens is that the line through A' parallel to BC, the line through B' parallel to CA and the line through C' parallel to AB are concurrent at a point P that always lies on circle A'B'C'. This result is proved below using Cartesian co-ordinates.

*CABRI II plus* indicates that if another point D is taken on circle ABC and its Simson line is drawn then PD' is perpendicular to the Simson line of D, where D' is the reflection of D in L. If D' coincides with P, then the tangent to circle A'B'C'at P is perpendicular to the Simson line of P.

Further, if L is dragged parallel to its initial position then there are two positions of L in which P also lies on circle ABC. In what follows the analytic condition for this to happen is obtained and comments from David Monk [1] as to their geometrical significance are described.

## 2. Choice of co-ordinates and the circle ABC

We take Cartesian co-ordinates with L the line y = 0 and points A, B, C with co-ordinates A(a, d), B(b, e), C(c, h). With these choices the equation of the circle ABC is

$$\{a(e-h) + b(h-d) + c(d-e)\}(x^2 + y^2) - (a^2(e-h) + b^2(h-d) + c^2(d-e) -$$
$$(h-d)(d-e)(e-h))x + (ab(a-b) + bc(b-c) + ca(c-a) - a(e^2-h^2) - b(h^2-d^2) - c(d^2-e^2))y$$
$$+ a^2(ce-bh) + b^2(ah-cd) + c^2(bd-ae) + aeh(e-h) + bhd(h-d) + cde(d-e) = 0. \quad (2.1)$$

## 3. The lines BC, CA, AB and the point P

The equations of the sides of the triangle ABC are as follows:

BC: $(h-e)x + (b-c)y = bh-ce,$ (3.1)

CA: $(d-h)x + (c-a)y = cd-ah,$ (3.2)

AB: $(e-d)x + (a-b)y = ae-bd.$ (3.3)

We choose the line L to have equation y = 0 so that the image triangle’s vertices have co-ordinates A'(a, – d), B'(b, – e), C'(c, – h).

The line through A' parallel to BC has equation

$$(h-e)x + (b-c)y = a(h-e) - d(b-c). \quad (3.4)$$

The line through B' parallel to CA has equation

$$(d-h)x + (c-a)y = b(d-h) - e(c-a). \quad (3.5)$$

The line through C' parallel to AB has equation

$$(e-d)x + (a-b)y = c(e-d) - h(a-b). \quad (3.6)$$

It may now be checked that these three lines concur at the point P with co-ordinates (x, y) where $x = (1/(a(e – h) + b(h – d) + c(d – e)))(a^2(e – h) + b^2(h – d) + c^2(d – e) + bc(e – h) + ca(h – d) + ab(d – e))$,

$$y = (1/(a(e – h) + b(h – d) + c(d – e)))(a(d – e – h)(e – h) + b(e – h – d)(h – d) + c(h – d – e)(d – e). \quad (3.7)$$

The circle A'B'C' has the same equation as that of ABC (Equation (2.1)) except that d, e, h must be replaced by – d, – e, – h respectively. And then it may be checked that P lies on circle A'B'C'.

## 4. When P lies on circle ABC

If one moves L (y = 0) parallel to itself then there are two cases when P lies also on circle ABC.

The first case is when P lies on L and then L is the common chord of circles ABC and A'B'C'. The analytic condition for this is

$$a(d – e – h)(e – h) + b(e – h – d)(h – d) + c(h – d – e)(d – e) = 0. \quad (4.1)$$

when y = 0 is the common chord.

The second case is when the line L passes through the centre O of circle ABC and then O' is the same point as O. The analytic condition for this is

$$(ab(a – b) + bc(b – c) + ca(c – a) – a(e^2 – h^2) – b(h^2 – d^2) – c(d^2 – e^2)) = 0. \quad (4.2)$$

David Monk [1] made a geometrical explanation of what is going on. I give his explanation, which is as follows:

Let L be the line in which one reflects: A to A' etc. Your first result, that the parallels to BC through A', to CA through B and to AB through C' meet in a point P, is very easy by coordinates; With L as x-axis A as (a,d), B(b,e), C(c,h) the concurrence of the parallels requires the vanishing of a 3x3 determinant which is immediate by adding rows.

Now for the question of when P lies on the circumcircle; this is very interesting. As a lucky guess I found that this is so when L is a diameter of the circumcircle. One suspects that there may be some connection with the orthopole and indeed this turns out to be so: If W is the orthopole of L with respect to ABC, H the orthocentre and P* the point diametrically opposite to the point of concurrence P then W (which is on the nine-point circle) is the midpoint of HP*.

But the diameters are not the only lines with the property. Let L* be the line parallel to L through P*; it also passes through P', the reflection of P in L. Then L* also has the property: indeed the point of concurrence is P' - on both the reflection line and the circle. All the above I established by complex numbers with ABC as unit circle.

*Reference*


1. D. Monk, private communication.



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