Article 71

More Special Cyclic Quadrilaterals

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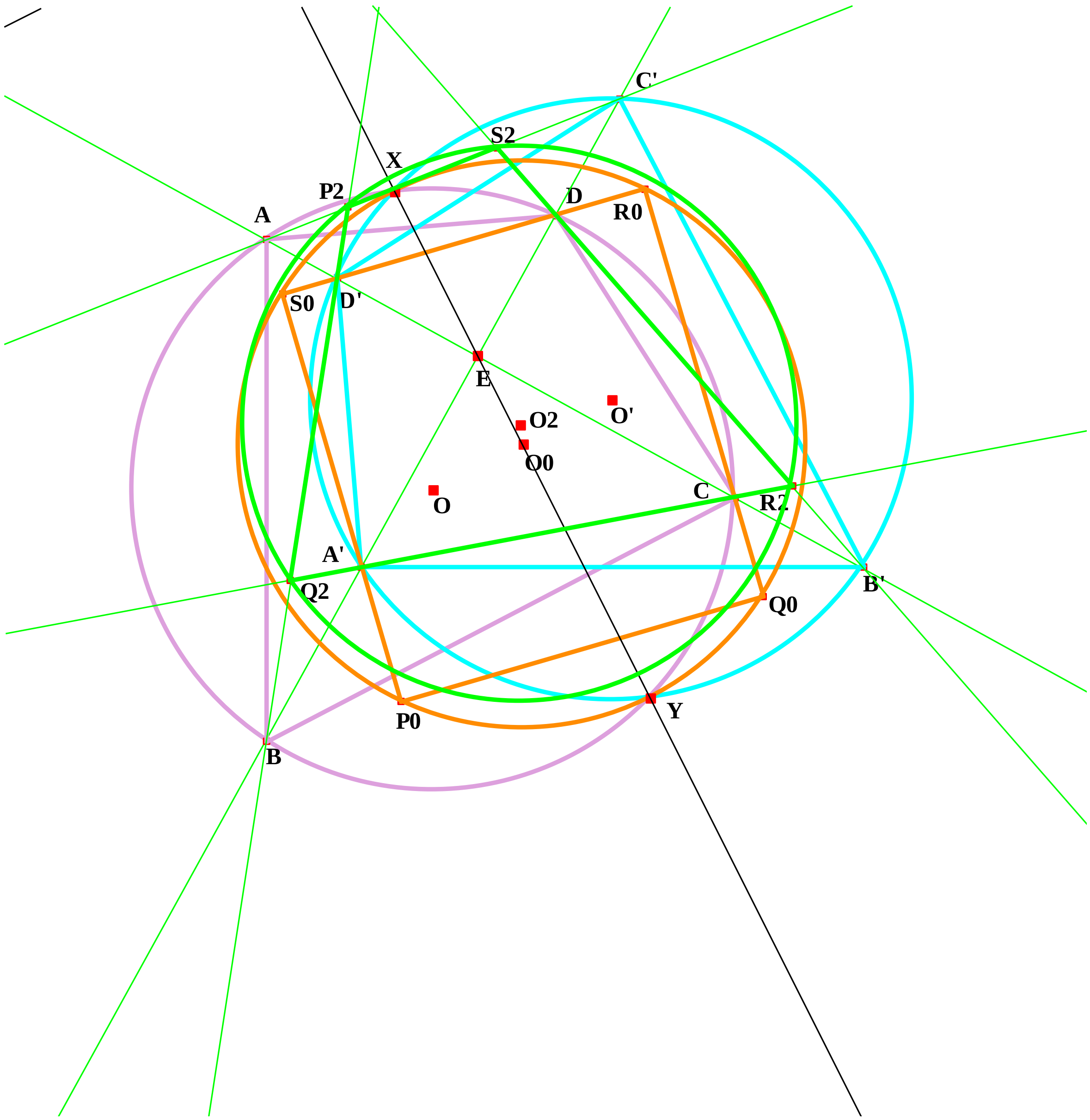

1. Introduction

This article is concerned with a cyclic quadrilateral ABCD in which AC is perpendicular to BD and its image A'B'C'D' after a rotation of 90^o about E = AC^BD. If AA'^BB' = P_0, BB'^CC' = Q_0, CC'^DD' = R_0, DD'^AA' = S_0, then $P_0Q_0R_0S_0$ is a rectangle and its circumcircle is coaxal with circles ABCD and A'B'C'D', all three meeting at points X, Y, where line XY passes through E. It is also the case that if AC'^BD' = P_2, BD'^CA' = Q_2,

$CA'\wedge DB' = R_2$, $DB'\wedge AC' = S_2$, then P_2, Q_2, R_2, S_2 are concyclic (but the circle is not part of the coaxal system). The centre O of ABCD, the centre O' of A'B'C'D' and E form three vertices of a square, with the centre O_0 of $P_0Q_0R_0S_0$ lying at the midpoints of the squares' diagonals. In the following sections these results are proved using Cartesian co-ordinates with E as origin.

2. The circles ABCD and A'B'C'D'

In choosing the co-ordinates of A, B, C, D we use the fact that EA.EC = EB.ED so with E as origin and EA along the positive direction of the x-axis we may set A(ab, 0), B(0, ac), C(– cd, 0), D(0, – bd), where a, b, c, d are positive real numbers.

Since A', B', C', D' are the images of A, B, C, D under a 90^o rotation centred on E we obtain for their co-ordinates A'(0, ab), B'(– ac, 0), C'(0, – cd), D'(bd, 0).

It is straightforward to find the equations of the circles ABCD and A'B'C'D' and these are

ABCD: $x^2 + y^2 + (cd - ab)x + (bd - ac)y - abcd = 0.$ (2.1)

A'B'C'D' $x^2 + y^2 + (ac - bd)x + (cd - ab)y - abcd = 0.$ (2.2)

These two circles have the same radius equal to $(1/2)\sqrt{\{(a^2 + d^2)(b^2 + c^2)\}}$, the centre O of circle ABCD has co-ordinates $((ab - cd)/2, (ac - bd)/2)$ and the centre O' of circle A'B'C'D' has co-ordinates $((bd - ac)/2, (ab - cd)/2)$.

3. The lines AA', BB', CC', DD' and the circle $P_0Q_0R_0S_0$

The lines AA', BB', CC', DD' have the following equations

AA': $x + y = ab,$ (3.1)

BB': $y - x = ac,$ (3.2)

CC': $x + y + cd = 0,$ (3.3)

DD': $x - y = bd.$ (3.4)

Clearly these lines form a rectangle. The co-ordinates of P_0, Q_0, R_0, S_0 are as follows:

$AA'\wedge BB' = P_0$: $(a(b - c)/2, a(b + c)/2).$ (3.5)

$BB'\wedge CC' = Q_0$: $(-c(a + d)/2, c(a - d)/2).$ (3.6)

$CC'\wedge DD' = R_0$: $(d(b - c)/2, -d(b + c)/2).$ (3.7)

$DD'\wedge AA' = S_0$: $(b(a + d)/2, b(a - d)/2).$ (3.8)

The circle $P_0Q_0R_0S_0$ has equation

$$2x^2 + 2y^2 + (a + d)(c - b)x + (d - a)(b + c)y - 2abcd = 0. \quad (3.9)$$

The square of its radius is $(1/8)\{(a^2 + d^2)(b^2 + c^2) + 4abcd\}$ and its centre O_0 has co-ordinates $((a + d)(b - c)/4, (a - d)(b + c)/4)$. It may now be checked that O_0 is the mid point of OO'. In fact OO' has equation

$$2(a + d)(b - c)x + 2(a - d)(b + c)y - (a^2 + d^2)(b^2 + c^2) + 4abcd = 0. \quad (3.10)$$

The line perpendicular to this through E has equation

$$(a + d)(b - c)y - (a - d)(b + c)x = 0 \quad (3.11)$$

and it may checked that O_0 lies on this line.

4. The points X and Y

We do not record the co-ordinates of both X and Y as it is the case that O_0 is the midpoint of XY and the co-ordinates involved are lengthy expressions.

X is defined as the point between A and D where EO_0 meets circle ABCD and its co-ordinates are (x, y) where

$$x = (a + d)(b - c)((a - d)\sqrt{((a^2 + d^2)(b^2 + c^2) + 4abcd)}\sqrt{((a^2 + d^2)(b^2 + c^2) - 4abcd)} + (a^3 - d^3)(b^2 + c^2) - ad(a - d)(b^2 + 4bc + c^2))/\{4(a - d)((a^2 + d^2)(b^2 + c^2) - 4abcd)\},$$

$$y = (b + c)((a - d)\sqrt{((a^2 + d^2)(b^2 + c^2) + 4abcd)}\sqrt{((a^2 + d^2)(b^2 + c^2) - 4abcd)} + (a^3 - d^3)(b^2 + c^2) - ad(a - d)(b^2 + 4bc + c^2))/\{4((a^2 + d^2)(b^2 + c^2) - 4abcd)\}. \quad (4.1)$$

It may now be checked that X and Y lie also on the circles centre O' and O_0, and hence that these three circles form part of an intersecting coaxal system.

5. The lines AC', BD', CA', DB' and the circle $P_2Q_2R_2S_2$

The equations of the four lines are

AC': $cdx - aby = abcd.$ (5.1)

BD': $acx + bdy = abcd.$ (5.2)

CA': $cdy - abx = abcd.$ (5.3)

DB': $-bdx - acy = abcd.$ (5.4)

The co-ordinates of the four points are as follows:

P_2 = AC'^BD' $\{ad/(a^2 + d^2)\}(b(a + d), c(d - a)).$ (5.5)

Q_2 = BD'^CA' $\{bc/(b^2 + c^2)\}(d(c - b), a(b + c)).$ (5.6)

R_2 = CA'^DB' $\{ad/(a^2 + d^2)\}(-c(a + d), b(d - a)).$ (5.7)

S_2 = DB'^ AC' $\{bc/(b^2 + c^2)\}(a(c - b), -d(b + c)).$ (5.8)

It may now be shown that these four points lie on a circle with equation

$$(a^2 + d^2)(b^2 + c^2)(x^2 + y^2) + (b - c)(bc(a^3 + d^3) - ad(a + d)(b^2 + bc + c^2))x - (b + c)(bc(a^3 - d^3) - ad(a - d)(b^2 - bc + b^2))y - (a^2 + d^2)(b^2 + c^2)abcd = 0. \quad (5.9)$$

The centre and radius of this circle are not worth recording, except to say that the centre does not lie on OO'.

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