Article 69

Two Cyclic Quadrilaterals and Two Coaxal Systems

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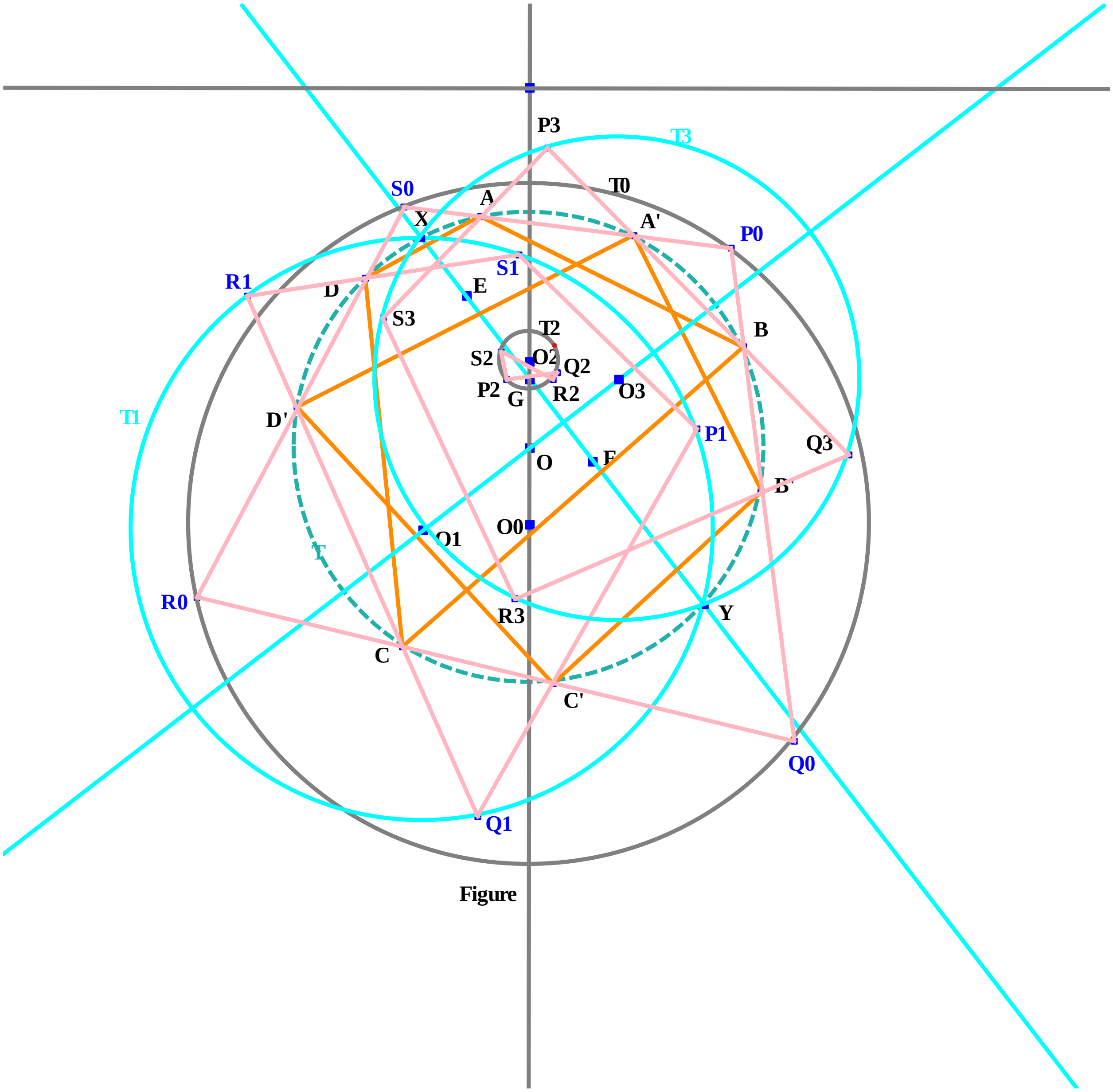

Figure

1. Introduction

Let ABCD and A'B'C'D' be two cyclic quadrilaterals inscribed in the same circle T, centre O, but having the following properties: (i) their diagonals are at right angles, the former at E and the latter at F; (ii) AC, BD are respectively parallel to A'C', B'D'. These conditions are essential for the consequential properties we now list. See the figure above.

(i) AA', BB', CC', DD' form a quadrilateral $T_0 = P_0Q_0R_0S_0$ (with AA' meeting BB' at P_0 etc.) which turns out to be cyclic, centre O_0;

(ii) AB', BC', CD', DA' form a quadrilateral $T_1 = P_1Q_1R_1S_1$(with AB' meeting BC' at P_1etc.) which again turns out to be cyclic, centre O_1;

(iii)AC', BD', CA', DB' form a quadrilateral $T_2 = P_2Q_2R_2S_2$(with AC' meeting BD' at P_2etc.) which again turns out to be cyclic, centre O_2;

(iv)AD', BA', CB', DC' form a quadrilateral $T_3 = P_3Q_3R_3S_3$(with AD' meeting BA' at P_2etc.) which again turns out to be cyclic, centre O_3;

(v) O, O_1, O_3 are collinear;

(vi)O, O_0, O_2, G are collinear, where G is the midpoint of EF;

(vii) T, T_1, T_3 are coaxal, with common chord EF;

(viii) T, T_0, T_2 are coaxal, with limiting point G.

2. Choice of axes and co-ordinates of points

We use Cartesian co-ordinates and because they meet at right-angles the best choice of axes are the internal bisectors of angles AOC and BOD, and these are also the internal bisectors of angles A'OC' and B'OD' since AC, BD are parallel to A'C'. B'D' respectively. An arbitrary point on circle T is taken to have co-ordinates $((1 - p^2)/(1+ p^2), 2p/(1 + p^2))$, where p is an appropriate parameter. The parameters are chosen as follows: A –t, B s, C t, D 1/s, A'– m, B' n, C' m, D' 1/n.

3. The circle T_0

The lines involved have equations

AA': $(1 - mt)x - (m + t)y = 1 + mt,$ (3.1)

BB': $(1 - sn)x + (s + n)y = 1 + sn,$ (3.2)

CC': $(1 - mt)x + (t + m)y = 1 + mt,$ (3.3)

DD': $-(1 - sn)x + (n + s)y = 1 + sn.$ (3.4)

Points involved have co-ordinates (x, y)

P_0 = AA'^BB':

$x = -(mns + mnt + mst + nst + m + n + s + t)/(mns + mnt + mst + nst - m - n - s - t),$

$y = 2(mt - ns)/(mns + mnt + mst + nst - m - n - s - t);$ (3.5)

Q_0 = BB'^CC':

$x = -(mns - mnt - mst + nst + m - n - s + t)/(mns - mnt - mst + nst - m + n + s - t),$

$y = 2(ns - mt)/(mns - mnt - mst + nst - m + n + s - t);$ (3.6)

R_0 = CC'^DD':
x = (mns – mnt – mst + nst +m – n – s + t)/(mns + mnt + mst + nst – m – n – s – t),
y = 2(mnst – 1))/(mns + mnt + mst + nst – m – n – s – t); (3.7)

S_0 = DD'^AA':
x = (mns + mnt +mst + nst + m + n + s + t)/(mns – mnt – mst + nst – m + n + s – t),
y = 2(1 – mnst)/(mns – mnt – mst + nst – m + n + s – t). (3.8)

It may now be checked that P_0, Q_0, R_0, S_0 lie on a circle T_0 with centre O_0 co-ordinates (u, v) where,
$u = (1 + n^2)(1 + s^2)(m^2t^2 – 1)/\{(mns + mnt + mst + nst – m – n – s – t)(mns – mnt – mst + nst – m + n + s – t)\}$,
$v = (1 + m^2)(1 + t^2)(n + s)(1 + ns)/\{(mns + mnt + mst + nst – m – n – s – t)(mns – mnt – mst + nst – m + n + s – t)\}$. (3.9)

4. The circle T_1

The lines involved have equations
AB': (1 + nt)x + (n – t)y = 1 – nt, (4.1)
BC': (1 – ms)x + (m + s)y = 1 + ms, (4.2)
CD': (n – t)x + (1 + nt)y = n + t, (4.3)
DA': (m + s)x + (1 – ms)y = s – m. (4.4)

Points involved have co-ordinates (x, y)
P_1 = AB'^BC':
x = – (mns + mnt – mst + nst – m + n – s – t)/(mns + mnt – mst + nst + m – n + s + t),
y = 2(ms + nt)/(mns + mnt – mst + nst + m – n + s + t); (4.5)

Q_1 = BC'^ CD':
x = – (mnst – mn + ms – mt + nt – ns – st + 1)/(mnst + mn + ms – mt + ns – nt – st – 1),
y = 2(mns – t)/(mnst + mn + ms – mt + ns – nt – st – 1); (4.6)

R_1 = CD'^ DA':
x = (mns – mnt + mst + nst – m – n + s – t)/(mns + mnt – mst + nst + m – n + s + t),
y = 2(mn + st)/(mns + mnt – mst + nst + m – n + s + t); (4.7)

S_1 = DA'^AB':
x = – (mnst + mn – ms – mt – ns – nt + st)/(mnst + mn + ms – mt + ns – nt – st – 1),
y = 2(m – nst)/(mnst + mn + ms – mt + ns – nt – st – 1). (4.8)

It may now be checked that P_1, Q_1, R_1, S_1 lie on a circle T_1 with centre O_1 co-ordinates (u, v) where,

$u = -(1 + m^2)(1 + t^2)(1 - sn)(s - n)/\{(mns + mnt - mst + nst + m - n + s + t)(mnst + mn + ms - mt + ns - nt - st - 1)\}$,

$v = (m^2 - t^2)(1 + n^2)(1 + s^2)/\{(mns + mnt - mst + nst + m - n + s + t)(mnst + mn + ms - mt + ns - nt - st - 1)\}$. (4.9)

It is not difficult now to show that the common chord of T and T_1 passes through E and F, where E has co-ordinates $((1 - t^2/(1 + t^2), 2s/(1 + s^2))$ and F has co-ordinates $((1 - m^2)/(1 + m^2), 2n/(1 + n^2))$ and the common chord has equation

$$(1 + m^2)(1 + t^2)(1 - sn)(s - n)x - (m^2 - t^2)(1 + n^2)(1 + s^2)y + m^2(n^2s(1 + t^2) + n(1 - t^2)(1 + s^2) + s(1 + t^2)) - n^2s(1 + t^2) + n(1 - t^2)(1 + s^2) - s(1 + t^2) = 0 \quad (4.10)$$

5. The circle T_2

The lines involved have the following equations

AC': $(1 + mt)x + (m - t)y = 1 - mt$, (5.1)

BD': $(n - s)x + (1 + ns)y = n + s$, (5.2)

CA': $(1 + mt)x + (t - m)y = 1 - mt$, (5.3)

DB': $(n - s)x - (1 + ns)y = -(n + s)$. (5.4)

Points involved have co-ordinates (x, y)

P_2= AC'^BD'

$x = -(mnst + mn + ms + mt - ns - nt - st - 1)/(mnst - mn + ms + mt + ns + nt - st + 1)$,

$y = 2(mnt + s)/(mnst - mn + ms + mt + ns + nt - st + 1)$; (5.5)

Q_2 = BD'^CA'

$x = -(mnst - mn - ms + mt + nt - ns + st - 1)/(mnst + mn - ms + mt + ns - nt + st + 1)$,

$y = 2(mnt + s)/(mnst + mn - ms + mt + ns - nt + st + 1)$; (5.6)

R_2= CA'^DB'

$x = -(mnst - mn - ms + mt + nt - ns + st - 1)/(mnst - mn + ms + mt + ns + nt - st + 1)$,

$y = 2(mst + n)/ (mnst - mn + ms + mt + ns + nt - st + 1)$; (5.7)

S_2 = DB'^CA'

$x = -(mnst +mn + ms + mt - ns - nt - st - 1)/(mnst + mn - ms + mt + ns - nt + st + 1)$,

$y = 2(mst + n)/(mnst + mn - ms + mt + ns - nt + st + 1)$. (5.8)

It may now be checked that P_2, Q_2, R_2, S_2 lie on a circle T_2, with centre O_2 having co-ordinates (u, v), where

$u = (1 + n^2)(1 + s^2)(1 - m^2t^2)/\{(mnst + mn - ms + mt + ns - nt + st + 1)(mnst - mn + ms$

+ mt + ns + nt – st + 1)},

$v = (1 + m^2)(1 + t^2)(1 + ns)(n + s)/\{(mnst + mn – ms + mt + ns – nt + st + 1)(mnst – mn + ms + mt + ns + nt – st + 1)\}$. (5.9)

It may also now be checked that O O0 and O O2 are the same line with equation

$$(1 + m^2)(1 + t^2)(1 + ns)(n + s)x = (1 + n^2)(1 + s^2)(1 – m^2t^2)y \qquad (5.10)$$

And that thus line passes through G, the midpoint of EF. It may also be shown, though we do not present details of the calculation, that circles T, T_0, T_2 are coaxal with limiting point G.

6. The circle T_3

The lines involved have the following equations

AD': $(n + t)x + (1 – nt)y = n – t$, (6.1)

BA': $(1 + ms)x + (s – m)y = 1 – ms$, (6.2)

CB': $(1 – nt)x + (n + t)y = 1 + nt$, (6.3)

DC': $(s – m)x + (1 + ms)y = s + m$. (6.4)

Points involved have co-ordinates (x, y)

P_3 = AD'^BA'

$x = – (mnst + mn – ms – mt – ns – nt + st +1)/(mnst – mn – ms – mt + ns + nt + st – 1)$,

$y = 2(t – mns)/(mnst – mn – ms – mt + ns + nt + st – 1)$; (6.5)

Q_3 = BA'^CB'

$x = – (mns – mnt + mst + nst – m – n + s – t)/(mns – mnt + mst + nst + m + n – s + t)$,

$y = 2(ms + nt)/(mns – mnt + mst + nst + m + n – s + t)$; (6.6)

R_3 = CB'^DC'

$x = – (mnst – mn + ms – mt + nt – ns – st + 1)/(mnst – mn – ms – mt + ns + nt + st – 1)$,

$y = 2(nst – m)/(mnst – mn – ms – mt + ns + nt + st – 1)$; (6.7)

S_3 = DC'^ AD'

$x = (mns + mnt – mst + nst – m + n – s – t)/(mns – mnt + mst + nst + m + n – s + t)$,

$y = 2(mn + st)/(mns – mnt + mst + nst + m + n – s + t)$. (6.8)

It may now be shown that P_3, Q_3, R_3, S_3 lie on the circle T_3, centre O_3 co-ordinates (u, v) where

$u = (1 + m^2)(1 + t^2)(s – n)(1 – sn)/\{(mns – mnt + mst + nst + m + n – s + t)(mnst – mn – ms – mt + ns + nt + st – 1)$,

$v = (t^2 – m^2)(1 + n^2)(1 + s^2)/\{(mns – mnt + mst + nst + m + n – s + t)(mnst – mn – ms – mt + ns + nt + st – 1)\}$. (6.9)

It may now be shown that O_3 lies on OO_1 and has common chord EF, showing that T, T_1, T_3 are coaxal.

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