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When I replaces K and Ge replaces H and Mi replaces O

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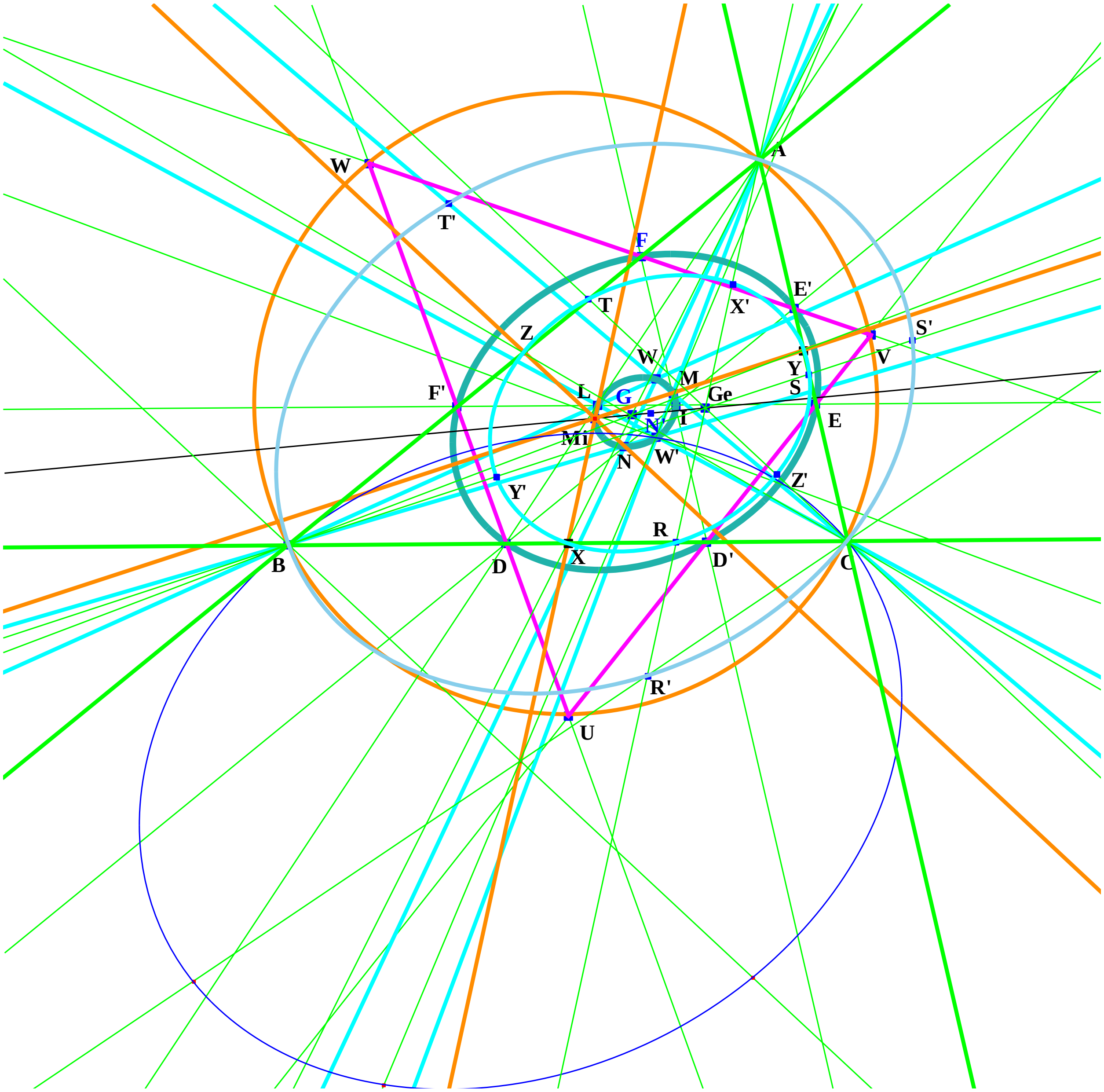

1. Introduction

A study is made of what happens when the incentre I with areal co-ordinates (a, b, c) replaces the symmedian point K with areal co-ordinates (a^2, b^2, c^2) in the construction of the Triplicate Ratio

Circle and the 7-point circle, and indeed in the circumcircle and the nine-point circle. Other points whose areal co-ordinates depend on a^2, b^2, c^2 only get replaced by those with identical functions of a, b, c. So, for example Gergonne's point Ge (1/(b + c – a), 1/(c + a – b), 1/(a + b – c)) replaces H ($1/(b^2 + c^2 - a^2)$, $1/(c^2 + a^2 - b^2)$, $1/(a^2 + b^2 - c^2)$) and the Mittenpunkt Mi replaces the circumcentre O.

Details are shown in the Figure and are explained more fully in the paragraphs that follow. An affine transformation is involved, so that midpoints remain as midpoints (and hence the centroid G and the midpoints of sides X, Y, Z remain unchanged). There is nothing really new in this article, but the various comparisons and constructions may well be a novelty to the reader. It is a classic example of how to obtain seemingly new results from those existing in the Euclidean plane.

2. The circumconic

Take lines AGe, BGe, CGe to met BC, CA, AB respectively in R, S, T and on these lines choose points R', S', T' so that GeR = RR', GeS = SS', GeT = TT'. The circumconic ABCR'S'T' may now be drawn and it has equation

$$ayz + bzx + cxy = 0. \qquad (2.1)$$

Its centre is the Mittenpunkt Mi with co-ordinates (a(b + c – a), b(c + a – b), c(a + b – c)).

Note that G remains in position with co-ordinates (1, 1, 1) and since Mi replaces O and Ge replaces H it follows that Mi, G, Ge are collinear and MiG = (1/3)MiGe.

3. The nine-point conic

In keeping with the classical construction the nine points on this conic are the feet R. S, T of the Cevians through Ge, the midpoints X, Y, Z of the sides BC, CA, AB respectively and the midpoints X', Y', Z' of the line segments AGe, BGe, CGe respectively. Its centre is the point N', the midpoint of MiGe.

4. The triplicate ratio conic

In the standard construction of the triplicate ratio circle one starts with the symmedian point K and draws the lines through K parallel to the sides. Here we use the incentre I, rather than K and then the lines through I parallel to the sides define the points D, D', E, E', F, F' with co-ordinates: D(0, a + b, c), D'(0, b, c + a), E(a, 0, b + c), E'(a + b, 0, c), F(c + a, b, 0), F'(a, b + c, 0). The conic through these six points is the triplicate ratio conic with equation

$$(b + c)x^2/a + (c + a)y^2/b + (a + b)z^2/c - yz(2 + a(a + b + c)/bc) - zx(2 + b(a + b + c)/ca) - xy(2 + c(a + b + c)/ab) = 0. \qquad (4.1)$$

If the lines F'D, D'E, E'F are drawn these lines define a triangle UVW such that U, V, W lie on the circumcircle of triangle ABC.

5. The 7-point conic

Lines through A, B, C parallel to the sides FD, DE, EF respectively concur at the Brocard like point W (1/b, 1/c, 1/a) and lines through A, B, C parallel to the sides D'E', E'F', F'D' concur at the Brocard like point W'(1/c, 1/a, 1/b). Let BW^CW' = L, CW^AW' = M, AW^BW' = N. L lies also on both MiX and F'E and has co-ordinates (a, c, b). M lies also on both MiY and D'F and has co-ordinates (c, b, a). N lies also on both MiZ and E'D and has co-ordinates (b, a, c).

The seven points Mi, W, W', L, M, N, I lie on the 7-point conic with the equation

$$bcx^2 + cay^2 + abz^2 - a^2yz - b^2zx - c^2xy = 0 \qquad (5.1)$$

The triplicate ratio conic and the 7-point conic have identical centres, the midpoint of IMi.

6. The Brocard like conics

It is well known that an additional property of the Brocard points Ω, Ω' is that there are equivalent defining properties involving there circles through Ω and three circles through Ω'. We show in the figure only one of these, which is an ellipse that touches AB at B and passes through C and W' and has its centre on the line LMiX. There are six of these ellipses, three passing through W and three passing through W'.

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