Article 60

Centroid-centred Similar Ellipses

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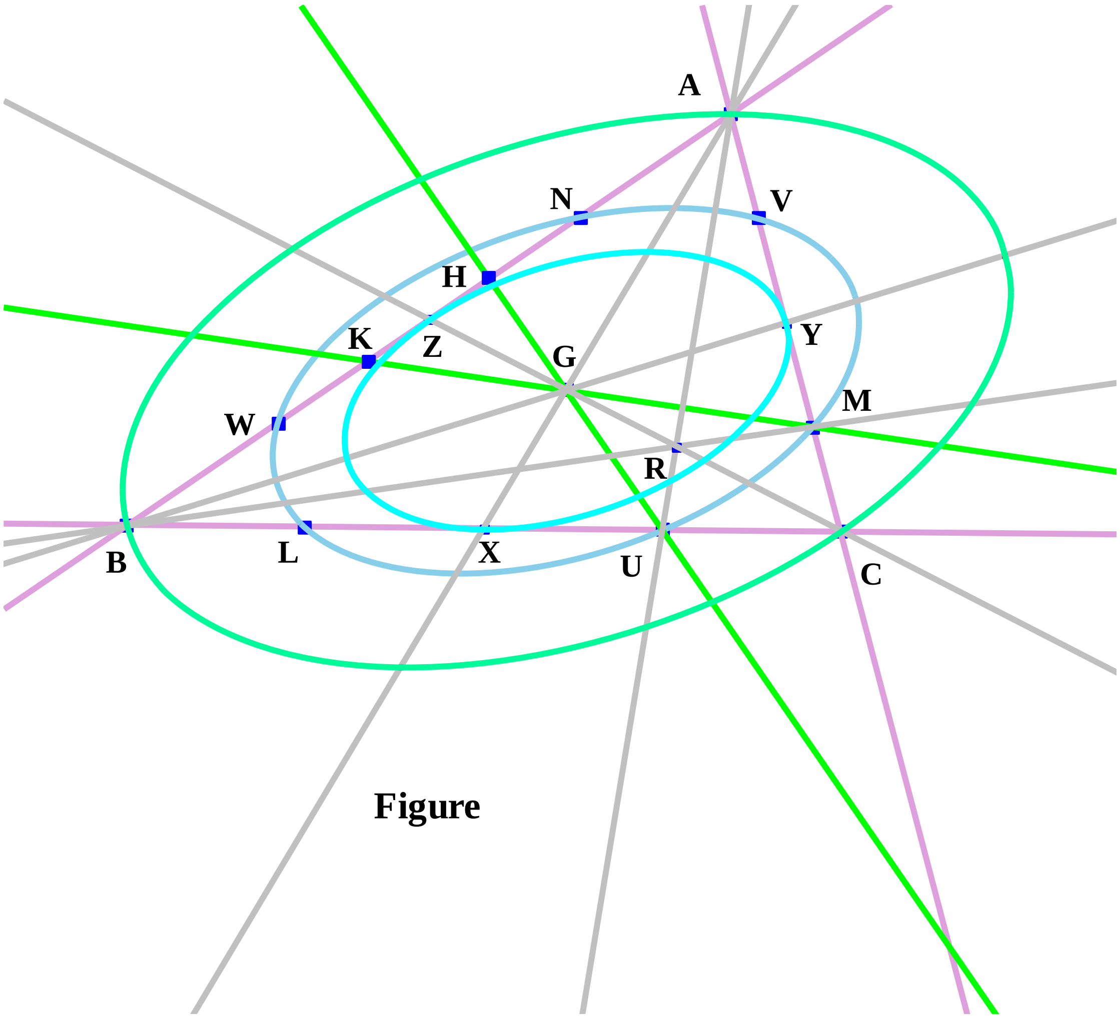

Figure

1. Introduction

The three ellipses drawn in the Figure are (i) the ellipse centre G passing through A, B, C known as the outer Steiner ellipse, (ii) the ellipse centre G touching the sides of ABC at its midpoints X, Y, Z and (iii) the intermediate ellipse centre G passing through points a quarter of the way along each side from each vertex, the points L, M, N, U, V, W. The co-ordinates in areals of some of the key points are listed and it is noted that AU, BM, CG amongst others are Cevian lines. It is also noted that if UG meets AB at H and MG meets AB at K, then HK is 1/5 AB. These ellipses are, of course, produced by an affinity from similar circles in an equilateral triangle.

2. The equations of the conics

The outer Steiner ellipse is well known to have an equation

$$yz + zx + xy = 0. \qquad (2.1)$$

It is the circumscribing conic of minimum area. It meets the circumcircle at the Steiner point whose co-ordinates are $(1/(b^2 – c^2), 1/(c^2 – a^2), 1/(a^2 – b^2))$.

The inner Steiner ellipse is well known to have the equation

$$x^2 + y^2 + z^2 – 2yz – 2zx – 2xy = 0. \qquad (2.2)$$

It is the inscribed conic of maximum area.

The inner and outer Steiner ellipses carry triangles that have an affinity with Poncelet’s porism.

The conic through the points L(0, 3, 1), M(1, 0, 3), N(3, 1, 0), U(0, 1, 3), V(3, 0, 1), W(1, 3, 0) has equation

$$3x^2 + 3y^2 + 3z^2 – 10yz – 10zx – 10xy = 0 \qquad (2.3)$$

For obvious reasons we call this the intermediate Steiner conic.

3. A few properties of the intermediate Steiner conic

The line UG has equation $3y = 2x + z$, so it meets $z = 0$ at the point H(3, 2, 0) so that AH = 2/5 AB.
The line UG meets the conic again at the point with co-ordinates (8, 5, – 1) and G(1, 1, 1) is the midpoint of this segment. It follows by a sequence of such arguments that G is the centre of the intermediate Steiner conic.
The line MG meets AB at a point K such that AK = 3/5 AB. It follows that HK = 1/5 AB and that the midpoint of HK is the midpoint Z of AB.
The lines AU, BM, CG meet at the point R (1, 1, 3). There are two other similar Cevian points with co-ordinates (3, 1, 1) and (1, 3, 1).
All three conics are similar as they arise from the same affinity.

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