Article 59

7 Points on any Circle not through a Vertex

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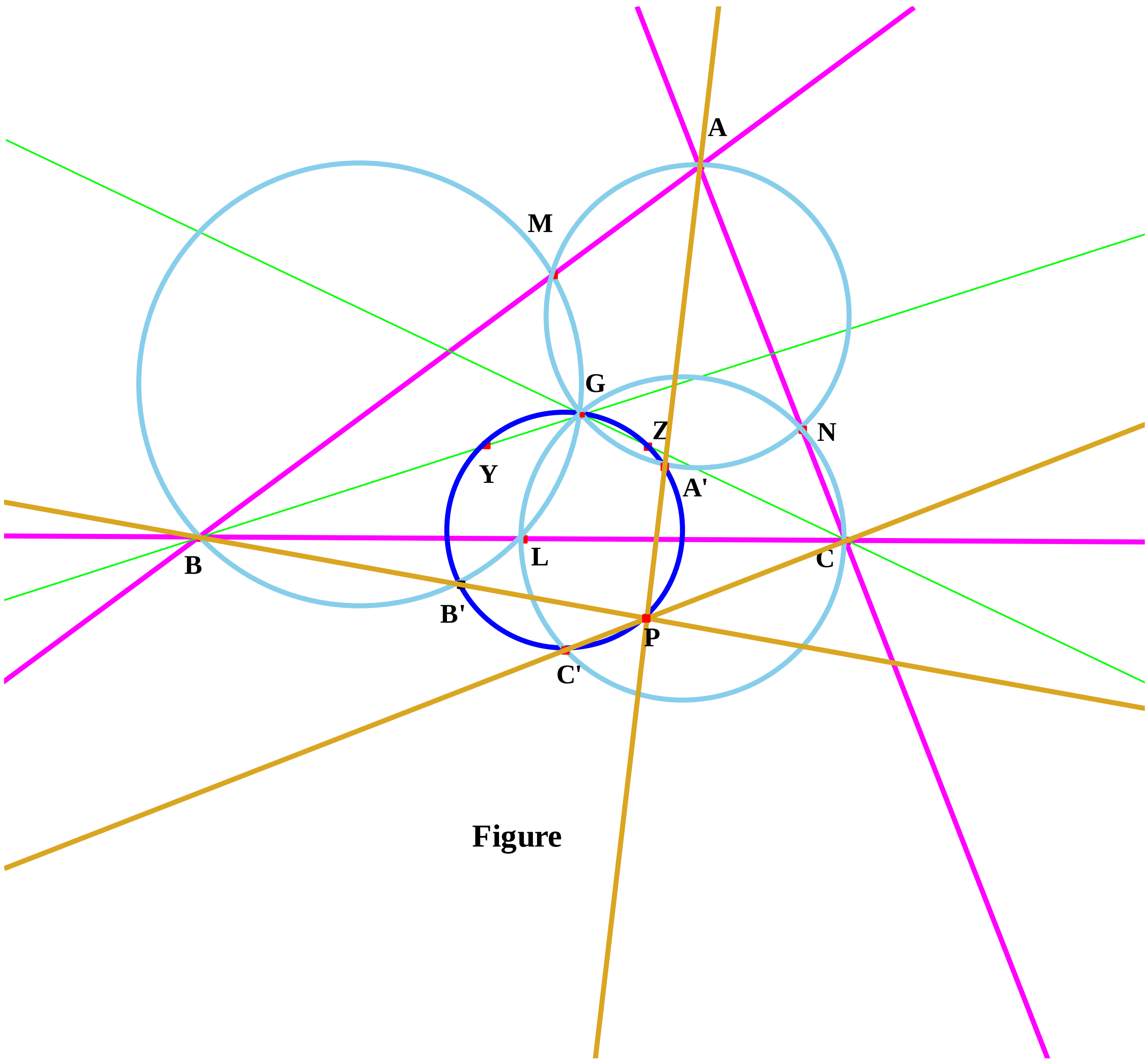

Figure

1. Introduction

The general result (of which a particular case is illustrated above and analysed below in later sections) is as follows: Choose a point Q not on the sides or extensions of the sides of a triangle (in the illustration the centroid G is chosen), then points L, M, N are chosen on the sides BC, CA, AB so that Q is the Miquel point for circles AMN, BNL, CLM. Any appropriate choice of points can be made. Any two points Y and Z are now chosen so that circle QYZ does not pass through a vertex (in the Figure Y and Z are such that BY = ¾ BG and CZ = ¾ CG). The three Miquel circles AMN, BNL, CLM now intersect circle QYZ again in points A', B', C' respectively. The beautiful result is that AA', BB', CC' are concurrent at a point P, which lies on circle QYZ. In what follows we establish the validity of the particular case, using areal co-ordinates. The general case is too difficult for *DERIVE.* It is hoped that a pure proof will emerge.

2. The circle GYZ

The points Y and Z are chosen to have co-ordinates (1, 2, 1) and (1, 1, 2) respectively, so that BY = ¾BG and CZ = ¾CG. The circle through these points and G(1, 1, 1) has equation

$$3(b^2 + c^2)x^2 + (2a^2 - b^2 + 2c^2)y^2 + (2a^2 + 2b^2 - c^2)z^2 - (8a^2 - b^2 - c^2)yz + (2a^2 - 7b^2 + 2c^2)zx + (2a^2 + 2b^2 - 7c^2)xy = 0. \qquad (2.1)$$

3. The circle BLG and the points B' and N

The choice of L is open to us and we choose it to lie at the midpoint of BC with co-ordinates (0, 1, 1). The equation of BLG now follows and is

$$(a^2 - 2b^2 - 2c^2)x^2 - 3a^2z^2 + 3a^2yz - 2(a^2 - 2b^2 + c^2)zx + (a^2 - 2b^2 + 4c^2)xy = 0. \qquad (3.1)$$

The circles GYZ and BLG meet at G and again at the point B' with co-ordinates (x, y, z, where

$$x = b^2 + c^2 - 2a^2,\quad y = 2c^2 + 2a^2 - b^2,\quad z = 3c^2. \qquad (3.2)$$

The circle BGL meets AB again at the point N with co-ordinates $(a^2 + 4c^2 - 2b^2, 2(b^2 + c^2) - a^2, 0)$.

4. The circle ANG and the points A' and M

Now that N is known we can derive the equation of circle ANG, which is

$$(a^2 - 2b^2 + 4c^2)y^2 + (a^2 + 4b^2 - 2c^2)z^2 - 2(2a^2 - b^2 - c^2)yz + (a^2 - 2b^2 - 2c^2)(zx + xy) = 0. \qquad (4.1)$$

Circle ANG and GYZ meet at G and again at the point A' with co-ordinates (x, y, z), where

$$x = 4b^2c^2 - b^4 - c^4,\quad y = b^2(b^2 + c^2),\quad z = c^2(b^2 + c^2). \qquad (4.2)$$

Circle ANG meets CA at M with co-ordinates $(a^2 - 2c^2 + 4b^2, 0, 2b^2 + 2c^2 - a^2)$.

5. **The circle CMG and the point C'**

Now that M is known we can derive the equation of circle CMG, which is

$$(a^2 - 2b^2 - 2c^2)x^2 - 3a^2y^2 + 3a^2yz + (a^2 + 4b^2 - 2c^2)zx - 2(a^2 + b^2 - 2c^2)xy = 0. \qquad (5.1)$$

This circle, by Miquel circle theory, must pass through L and this has been checked. It also meets circle GYZ at the point C' with co-ordinates $(b^2 + c^2 - 2a^2, 3b^2, 2a^2 + 2b^2 - c^2)$.

6. **AA', BB', CC' meet at a point P lying on circle GYZ**

It is now straightforward to show that AA', BB', CC' are concurrent at a point P whose co-ordinates are $(b^2 + c^2 - 2a^2, 3b^2, 3c^2)$ and that P lies on circle GYZ.

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