Article 52

Three triangles in mutual Triple Reverse Perspective

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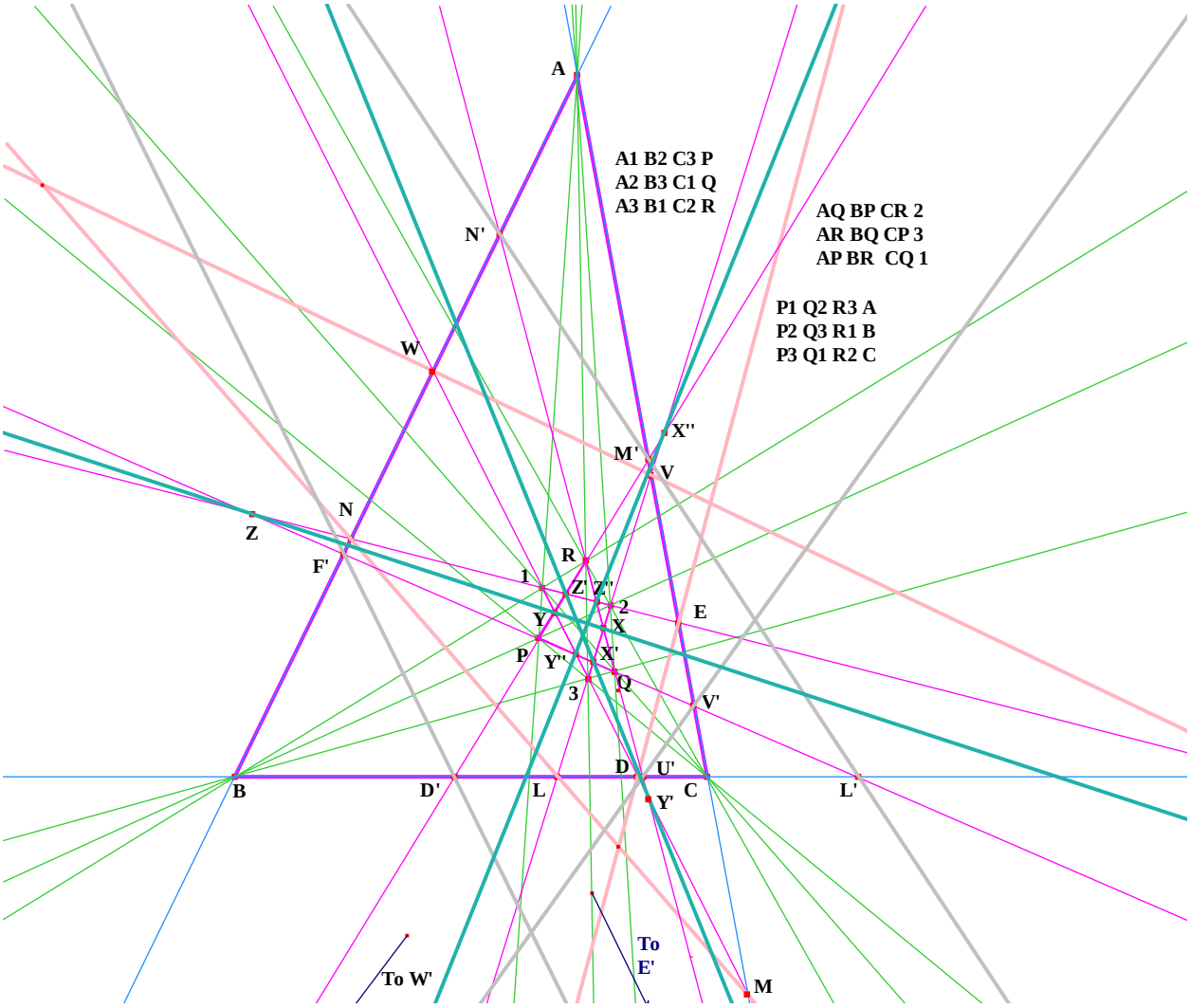

1. Introduction

The three triangles concerned are triangles ABC, PQR and 132 (all having anticlockwise labelling). The figure is interesting in the sense that P, Q, R are the perspectors of triangles ABC and 123, that 1, 2, 3 are the perspectors of triangles ABC and QPR and that A, B, C are the perspectors of triangles PQR and 123. (*Apologies to those who feel that labels 2 and 3 ought to*

have been interchanged, but that would have meant a rewriting of the analysis and that might well have introduced errors.)

Note how the figure is drawn in order to provide total accuracy. First triangle ABC and points P and Q were chosen. Then lines AP, BP, CP, AQ, BQ, CQ were drawn. Now point 1 was defined as AP^CQ and point 2 was defined as AQ^BP. Then point 3 was defined as CP^BQ. Lines B1, C2, A3 are then found to be concurrent at the point R.

The method of construction ensures the mutual triple reverse perspectives mentioned above. No further analysis is actually needed to establish this, but in view of intended applications it becomes necessary to set up the analysis and to have recorded the co-ordinates of all points and the equations of the nine perspectrices. Areal co-ordinates are used with ABC as triangle of reference.

2. Points P, Q, R, 1, 2, 3

We take P to have co-ordinates (p, q, r) and Q to have co-ordinates (l, m, n). Then lines AP, BP, CP have equations ry = qz, rx = pz and qx = py respectively. LinesAQ, BQ, CQ have equations ny = mz, nx = lz and mx = ly respectively.

AP meets CQ at point 1(lq, mq, mr). BP meets AQ at point 2(np, mr, nr). CP meets BQ at point 3(lp, lq, np).

B1 has equation mrx = lqz, A3 has equation npy = lqz and C2 has equation mrx = npy. R is the point of concurrence of these lines and has co-ordinates (lqnp, lqmr, mrnp).

3. Lines 12, 23, 31 and Desargues' axes LMN, DEF, UVW

Line 12 has equation

$$mr(mr - nq)x + nr(lq - mp)y + mq(np - lr)\,z = 0. \qquad (3.1)$$

Line 23 has equation

$$nr(lq - mp)x - pn(lr - np)y - pl(nq - mr)z = 0. \qquad (3.2)$$

Line 31 has equation

$$mq(lr - np)x - lp(mr - nq)y - lq(lq - mp)z = 0. \qquad (3.3)$$

Now A1, B2, C3 meet at P and the Deasargues' axis of this perspective is LMN, where L = BC^23, M = CA^31, N = AB^12. L has co-ordinates (0, l(mr – nq), n(lr – np)). M has co-

ordinates (l(mp – lq), 0, m(np – lr)). N has co-ordinates (n(lq – mp), m(nq – mr), 0). The axis LMN has equation

$$mx/(lq - mp) + ny/(mr - nq) + lz/(np - lr) = 0. \quad (3.4)$$

Triangles ABC and 231 are in perspective with vertex Q and the Desargues' axis of perspective is DEF, where D = BC^31, E = CA^12, F = AB^23. D has co-ordinates (0, q(lq – mp), p(nq – mr)). E has co-ordinates (q(lr – np), 0, r(mr – nq)). F has co-ordinates (p(lr – np), r(lq – mp), 0). The axis DEF has equation

$$rx/(lr - np) + py/(mp - lq) + qz/(nq - mr) = 0. \quad (3.5)$$

Triangles ABC and 312 are in perspective with vertex R and the Desargues' axis of perspective is UVW, where U = BC^12, V = CA^23, W = AB^31. U has co-ordinates (0, mq(lr – np), nr(lq – mp)). V has co-ordinates (lp(mr – nq), 0, nr(mp – lq)). W has co-ordinates (lp(mr – nq), mq(lr – np), 0). The axis UVW has equation

$$x/(lp(mr - nq)) + y/(mq(np - lr)) + z/(nr(lq - mp)) = 0. \quad (3.6)$$

4. Lines PQ, QR, RP and axes D'E'F', L'M'N', U'V'W'

The line PQ has equation

$$(nq - mr)x + (lr - np)y + (mp - lq)z = 0. \quad (4.1)$$

The line QR has equation

$$mnr(lq - mp)x + lnp(mr - nq)y + lmq(np - lr)z = 0. \quad (4.2)$$

The line RP has equation

$$mqr(lr - np)x + pnr(mp - lq)y + plq(nq - mr)z = 0. \quad (4.3)$$

Triangles ABC and QPR are in perspective with vertex 2 and the Desargues' axis of perspective is D'E'F' where D' = BC^PR, E' = CA^RQ, F' = AB^QP. D' has co-ordinates (0, lq(nq – mr), nr(lq – mp)). E' has co-ordinates (lq(lr – np), 0, nr(lq – mp)). F' has co-ordinates (lr – np, mr – nq, 0). The axis D'E'F' has equation

$$nrx/(lr - np) + nry/(nq - mr) + lqz/(mp - lq) = 0. \quad (4.4)$$

Triangles ABC and RQP are in perspective vertex 3 and the Desargues' axis of perspective is L'M'N', where L' = BC^QP, M' = CA^ PR, N' = AB^RQ. L' has co-ordinates (0, lq – mp, lr – np). M' has co-ordinates (lp(mr – nq), 0, mr(lr – np)). N' has co-ordinates (lp(nq – mr), mr(lq – mp), 0). The axis L'M'N' has equation

$$mrx/(mr - nq) + lpy/(lq - mp) + lpz/(np - lr) = 0. \quad (4.5)$$

Triangles ABC and PRQ are in perspective with vertex 1 and the Desargues' axis of perspective is U'V'W', where U' = BC^RQ, V' = CA^QP, W' = AB^PR. U' has co-ordinates (0, mq(lr – np), np(mr – nq)). V' has co-ordinates (lq – mp, 0, nq – mr). W' has co-ordinates (np(lq – mp), mq(lr – np), 0). The axis U'V'W' has equation

$$mqx/(lq - mp) + npy/(np - lr) + mqz/(mr - nq) = 0. \qquad (4.6)$$

5. Axes XYZ, X'Y'Z', X''Y''Z''

Triangles PQR and 123 are in perspective with vertex A and the Desargues' axis of perspective is XYZ, where X = QR^23, Y = RP^31, Z = PQ^12. X has co-ordinates ($lp(lmr^2 - n^2pq)$, $lmr(lqr + mpr - 2npq)$, $npr(2lmr - lnq - mnp)$). Y has co-ordinates ($lp(m^2pr - lnq^2)$, $lmq(2mpr - npq - lqr)$, $mpr(lmr + mnp - 2lnq)$). Z has co-ordinates ($l^2qr - mnp^2$, $m(2lqr - mpr - npq)$, $r(lmr + lnq - 2mnp)$). The axis XYZ has equation

$$mr((l^2qr(mr + nq) + lp(m^2r^2 - 6mnqr + n^2q^2) + mnp^2(mr + nq))x - pr(l^2(m^2r^2 - mnqr + n^2q^2) - lmnp(mr + nq) + m^2n^2p^2)y - lm(l^2q^2r^2 - lpqr(mr + nq) + p^2(m^2r^2 - mnqr + n^2q^2))z = 0. \qquad (5.1)$$

Triangles PQR and 231 are in perspective with vertex B and the Desargues' axis of perspective is X'Y'Z', where X' = PQ^23, Y'= QR^31, Z' = RP^12. X' has co-ordinates ($p(lmr + mnp - 2lnq)$, $m^2pr - lnq^2$, $n(2mpr - npq - lqr)$). Y' has co-ordinates ($lpq(lmr + lnq - 2mnp)$, $mq(l^2qr - mnp^2)$, $mnp(2lqr - mpr - npq)$). Z' has co-ordinates ($npq(2lmr - lnq - mnp)$, $mq(lmr^2 - n^2pq)$, $mnr(lqr + mpr - 2npq)$). The axis X'Y'Z' has equation

$$mn(l^2q^2r^2 - lpqr(mr + nq) + p^2(m^2r^2 - mnqr + n^2q^2))x - pn(l^2qr(mr + nq) + lp(m^2r^2 - 6mnqr + n^2q^2))y + pq(l^2(m^2r^2 - mnqr + n^2q^2) - lmnp(mr + nq) + m^2n^2p^2)z = 0. \qquad (5.2)$$

Triangles PQR and 312 are in perspective with vertex C and the Desargues' axis of perspective is X"Y"Z" where X" = PR^32, Y" = PQ^31, Z"= QR^12. X" has co-ordinates ($lnp(2lqr - pmr - pnq)$, $lqr(lmr + lnq - 2mnp)$, $nr(l^2qr - mnp^2)$). Y" has co-ordinates ($l(lqr + pmr - 2npq)$, $q(2lmr - lnq - mnp$, $lmr^2 - n^2pq)$. Z" ha co-ordinates ($lnq(lqr + npq - 2mpr)$, $mqr(2lnq - lmr - mnp)$, $nr(lnq^2 - m^2pr)$. The axis X"Y"Z" has equation

$$qr(l^2(m^2r^2 - mnqr + n^2q^2) - lmnp(mr + nq) + m^2n^2p^2)x + ln(l^2q^2r^2 - lpqr(mr + nq) + p^2(m^2r^2 - 6mnqr + n^2q^2))y - lq(l^2qr(mr + nq) + lp(m^2r^2 - 6mnqr + n^2q^2) + mnp^2(mr + nq))z = 0. \qquad (5.3)$$

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