# Perpendicular Bisectors and Angle Bisectors

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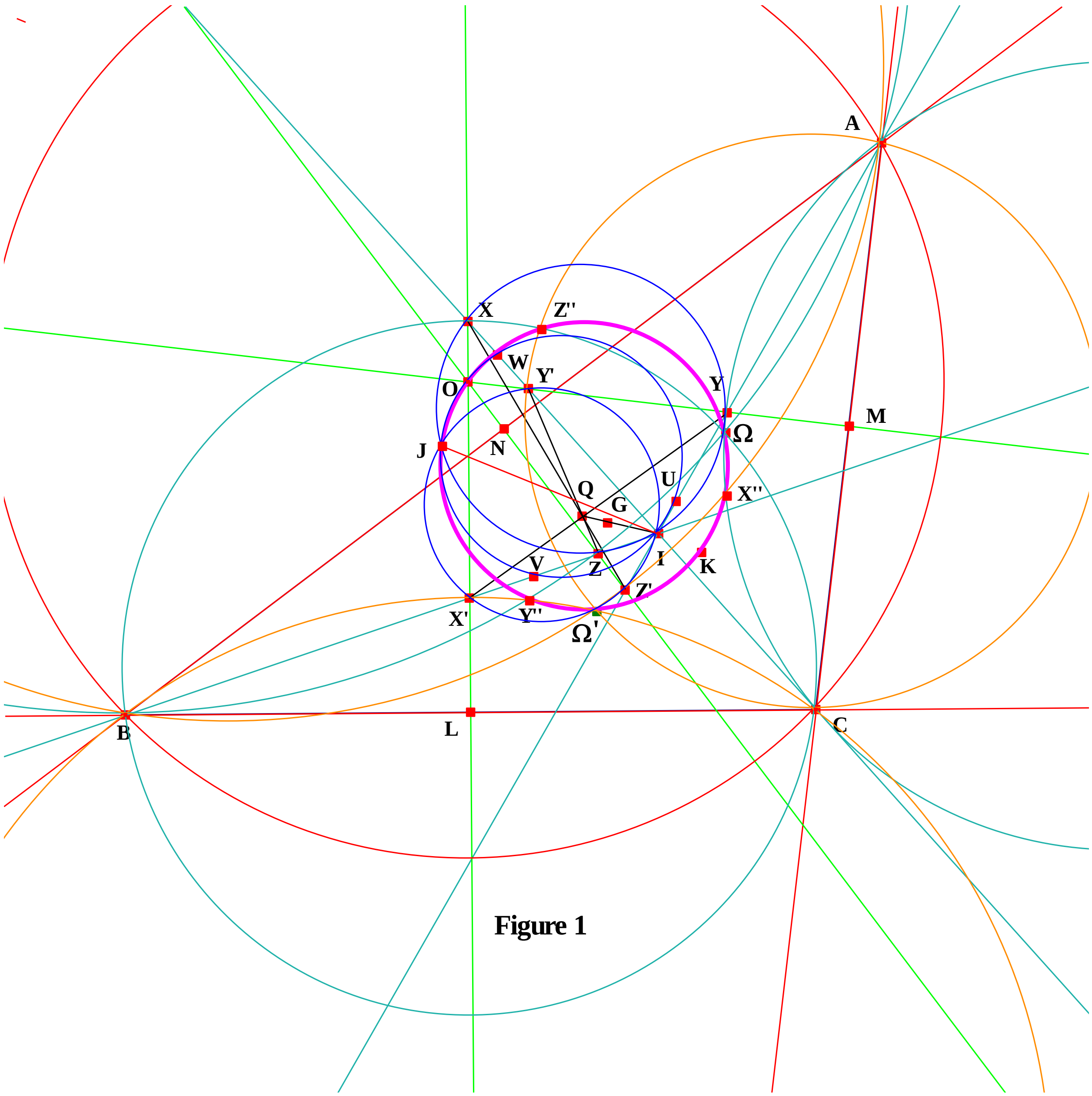



**Figure 1**

## 1. Introduction

In triangle ABC the incentre is I and the circumcentre is O. The angle bisectors AI, BI, CI are drawn, as well as the perpendicular bisectors of the sides OL, OM, ON. Of the nine intersections of these pairs of three lines, three, namely AI^OL, BI^OM, CI^ON are on the circumcircle (at the midpoints of the minor arcs). The other six are labeled as follows AI^OM = Y, BI^ON = Z, CI^OL = X, OL^BI = X', OM^CI = Y', ON^AI = Z'. Circles XYZ and X'Y'Z' are drawn, and for reasons that will shortly become clear the 7-point circle is also drawn.

The following results now hold:

(i) Circles XYZ and X'Y'Z' both pass through I, and they pass through the Brocard points Ω and Ω' respectively;

(ii) Their second point of intersection J lies on the 7-point circle;

(iii) The line IJ passes through K, the symmedian point of ABC;

(iv) Triangles YZX and Z'X'Y' are similar;

(v) Lines YX', ZY', XZ' are concurrent at Q, the Spieker centre of ABC;

(vi) If U, V, W are the midpoints of YZ', ZX', XY' respectively then circle UVW has OI as a diameter and passes through J;

(vii) Circles AYC, BZA, CXB pass through the Brocard point Ω and circles AY'C, BZ'A, CX'B pass through the Brocard point Ω';

(viii) Circle AYΩC and BZ'Ω'A meet at a point X" lying on the 7-pt circle, circle BZΩA and CX'Ω'B meet at a point Y" lying on the 7-pt circle and circle CXΩB and AY'Ω'C meet at a point Z" also lying on the 7-pt circle;

(ix) Circles YZZ', ZXX', XYY' have a common point S' lying on circle X'Y'Z';

(x) Circles X'Y'X, Y'Z'Y, Z'X'Z have a common point S lying on circle XYZ;

(xi) Circles AYZ, BZX, CXY have a common point D lying on the circumcircle, circles AY'Z', BZ'X', CX'Y' have a common point D' lying on the circumcircle and circles AY"Z", BZ"X", CX"Y" have a common point D" lying on the circumcircle.

(xii) Circles BX"C, CY"A, AZ"B all pass through O.

The line of centres of circles XYZ and X'Y'Z' is, of course perpendicular to IJ through its midpoint, but the centres do not appear to be significant. Results (i) to (viii) are illustrated in Figure 1 and results (ix) to (xii) are illustrated in Figure 2. Results are proved in the following sections using areal-co-ordinates with ABC as triangle of reference.

## 2. The angle bisectors and perpendicular bisectors

The equations of the angle bisectors AI, BI, CI are respectively cy = bz, az = cx, bx = ay.

The equation of the perpendicular bisector of BC is the line OL, with equation

$$(b^4 - c^4 - a^2b^2 + a^2c^2)x + a^2(b^2 + c^2 - a^2)(y - z) = 0. \qquad (2.1)$$

The equations of the perpendicular bisectors of CA, AB may be obtained from Equation (2.1) by cyclic change of x, y, z and a, b, c,.

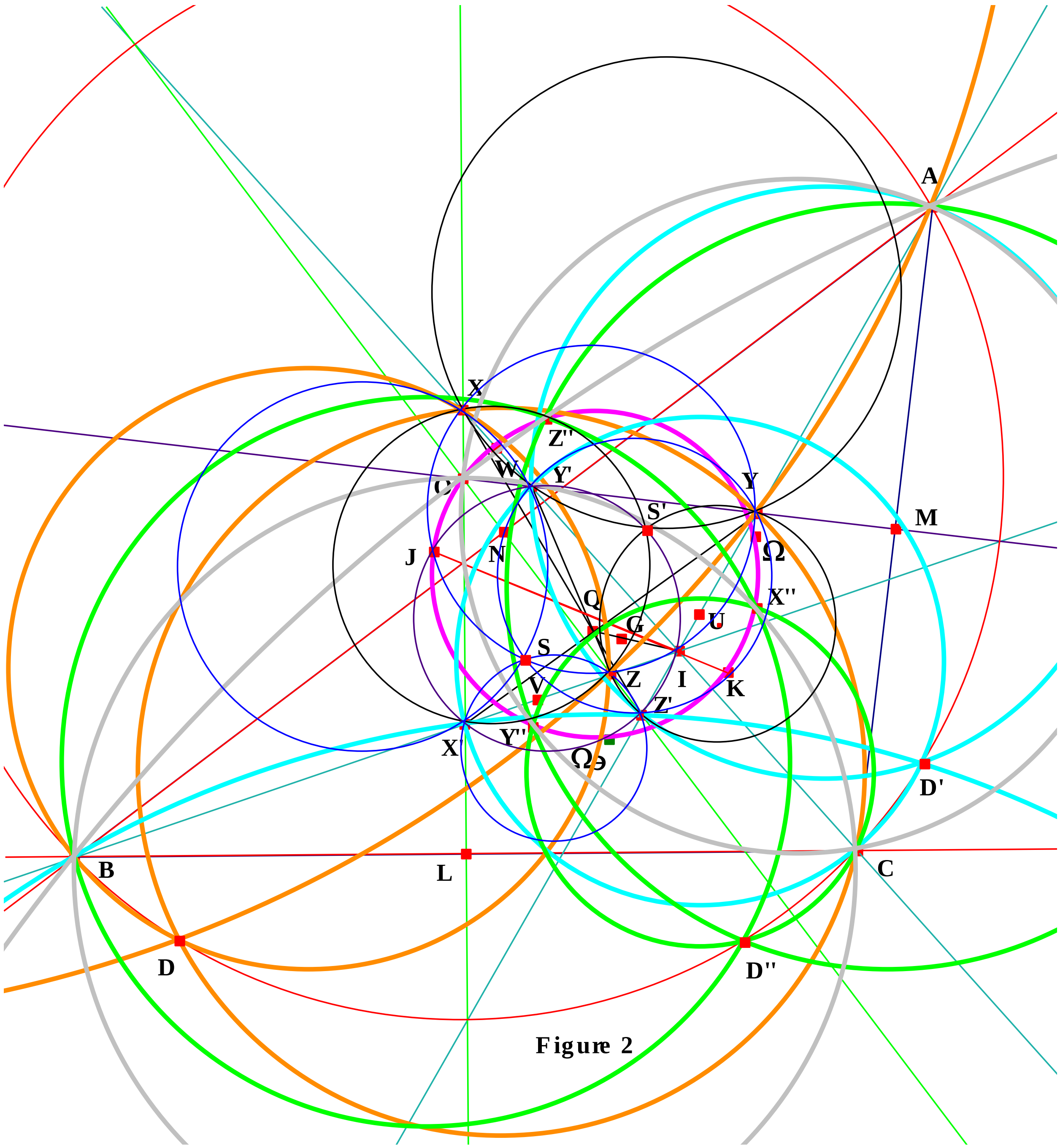



Figure 2

### 3. The six points X, Y, Z, X', Y', Z'

The co-ordinates of these six points are as follows:

$$Y(bc + c^2 - a^2, b^2, bc),\ Z(ca, ca + a^2 - b^2, c^2),\ X(a^2, ab, ab + b^2 - c^2),$$
$$X'(a^2, ca + c^2 - b^2, ca),\ Y'(ab, b^2, ab + a^2 - c^2),\ Z'(bc + b^2 - a^2, bc, c^2).$$

## 4. The circles XYZ and X'Y'Z' and their common chord IJ

The circle XYZ has equation

$$b^2cx^2 + c^2ay^2 + a^2bz^2 - a(ca + a^2 - c^2)yz - b(ab + b^2 - a^2)zx - c(bc + c^2 - b^2)xy = 0. \quad (4.1)$$

It may now be checked that I(a, b, c) lies on this circle and also that it passes through the Brocard point $\Omega(1/b^2, 1/c^2, 1/a^2)$.

The circle X'Y'Z' has equation

$$bc^2x^2 + ca^2y^2 + ab^2z^2 - a(ab + a^2 - b^2)yz - b(bc + b^2 - c^2)zx - c(ca + c^2 - a^2)xy = 0. \quad (4.2)$$

It may now be checked that I(a, b, c) lies on this circle and also that it passes through the Brocard point $\Omega'(1/c^2, 1/a^2, 1/b^2)$.

The second point of intersection of the two circles is J, which has co-ordinates

$$\begin{aligned} x &= a(a^4 - a^3(b + c) - a^2bc + 2abc(b + c) - bc(b^2 + c^2)),\\ y &= b(b^4 - b^3(c + a) - b^2ca + 2abc(c + a) - ca(c^2 + a^2)),\\ z &= c(c^4 - c^3(a + b) - c^2ab + 2abc(a + b) - ab(a^2 + b^2). \end{aligned} \quad (4.3)$$

It may now be checked that J lies on the 7-point circle with equation

$$b^2c^2x^2 + c^2a^2y^2 + a^2b^2z^2 - a^4yz - b^4zx - c^4xy = 0. \quad (4.4)$$

The equation of IJ is

$$bc(b - c)x + ca(c - a)y + ab(a - b)z = 0. \quad (4.5)$$

It is immediate that the line IJ passes through the symmedian point $K(a^2, b^2, c^2)$.

## 5. The similarity of triangles YZX and Z'X'Y'

The circles YZX and Z'X'Y' intersect at I and J, and YZ', ZX', XY' all pass through I. It follows from Wood [1], that triangle Z'X'Y' is similar to YZX by means of a direct similarity composed of a rotation about J followed by an enlargement.

## 6. The concurrence of YX', ZY', XZ' at the Spieker centre Q

The equation of YX' is

$$b(b - c)x + a(a - c)y - (a^2 + b^2 - ab - c^2)z = 0. \quad (6.1)$$

The equations of ZY' and XZ' follow from Equation (6.1) by cyclic change of x, y, z and a, b, c. These three lines all pass through Q(b + c, c + a, a + b), which is the Spieker centre lying on line IG, where G is the centroid of ABC and IG = 2 GQ.

## 7. Points U, V, W and properties of circle UVW

U is the midpoint of YZ' and so has co-ordinates $U((b + c)^2 - 2a^2, b(b + c), c(b + c))$. Similarly V has co-ordinates $V(a(c + a), (c + a)^2 - 2b^2, c(c + a))$ and W has co-ordinates $W(a(a + b), b(a + b), (a + b)^2 - 2c^2)$.

The equation of the circle UVW may now be obtained and it has equation

$$bc(b + c)x^2 \; ca(c + a)y^2 + ab(a + b)z^2 - a(2a^2 - b^2 - c^2 + a(b + c))yz$$
$$- b(2b^2 - c^2 - a^2 + b(c + a))zx - c(2c^2 - a^2 - b^2 + c(a + b))xy = 0. \quad (7.1)$$

It now follows that this circle passes through O and I and has centre the midpoint of OI and furthermore passes through J.

## 8. Circles YZZ', ZXX' and XYY' and the point S' lying on circle X'Y'Z'

The equation of the circle YZZ' is

$$b^2c^2x^2 + c^2(c(a + b) - a^2)y^2 + bc(a^2 + b^2 - ca)z^2 + (a^4 - a^2(b^2 + bc + 2c^2) + ac^2(c - b) +$$
$$bc(b^2 + c^2))yz + b(a^2(b + c) - ac^2 - b^2(b + c))zx + c^3(a - b - c)xy = 0. \quad (8.1)$$

Circles ZXX' and XYY' have equations that may be written down from Equation (8.1) by cyclic change of x, y, z and a, b, c.

It may now be verified that these circles have a common point S' with co-ordinates (x, y, z), where

$$x = a(a^3(b + c) - a^2(b + c) - abc(b + 2c) + b^2c(b - c)),$$
$$y = - b(a^2(b + c)^2 - a(b^3 - bc^2 + c^3) - b^2c(b - c)), \quad (8.2)$$
$$z = c(a^3b - a^2(b^2 + bc + c^2) + ac(c^2 - 2b^2) - bc^2(b - c)).$$

It may now be checked that S' lies on circle X'Y'Z'. It follows from Result 2 in the Appendix that circles X'Y'Z, Y'Z'X, Z'X'Y have a common point S lying on circle XYZ.

## 9. The connection with the Brocard points

Circle AYC has equation

$$c^2y^2 + (c^2 - a^2)yz - b^2zx = 0. \quad (9.1)$$

It may now be checked that this circle passes through the Brocard point Ω $(1/b^2, 1/c^2, 1/a^2)$. Similarly circles BZA and CXB pass through Ω. These three circles are, of course, the circles that are often chosen to define Ω. Their appearance in the context of the angle bisectors is mildly surprising.

Circle AY'C has equation

$$a^2y^2 - b^2zx + (a^2 - c^2)xy = 0. \quad (9.2)$$

This circle passes through the other Brocard point Ω' $(1/c^2, 1/a^2, 1/b^2)$, as do the circles BZ'A and CX'B.

The equation of the circle BZ'A is

$$b^2z^2 + (b^2 - a^2)yz - c^2xy = 0. \quad (9.3)$$

Circles AYΩC and BZ'Ω'A meet at X" with co-ordinates $(b^2 + c^2 - a^2, b^2, c^2)$. Similarly circles BZΩA and CX'Ω'B meet at Y' with co-ordinates $(a^2, c^2 + a^2 - b^2, c^2)$ and circles CXΩB and AY'Ω'C meet at Z" with co-ordinates $(a^2, b^2\ a^2 + b^2 - c^2)$. Points X", Y", Z" are the vertices of the second Brocard triangle and lie on the 7-point circle with Equation (4.4).

## 10. The points D, D', D''

It has already been established that circles AYC, BZA, CXB pass through Ω, as does circle XYZ. It follows from Result 2 of the Appendix that circles AYZ, BZX, CXY meet at a point D on the circumcircle of ABC. Similarly circles AY'C, BZ'A, CX'B pass through Ω', as does circle X'Y'Z'. It follows again from Result 2 of the Appendix that circles AY'Z', BZ'X', CX'Y' meet at a point D' on the circumcircle of ABC.

The equation of the circle CX"Y" is

$$b^2c^2(b^2 - c^2)x^2 - c^2a^2(c^2 - a^2)y^2 - a^2(a^4 - a^2b^2 - a^2c^2 + b^4)yz - b^2(a^4 - a^2b^2 + b^4 - b^2c^2)zx + c^2(a^2 - c^2)(b^2 - c^2)xy = 0. \quad (10.1)$$

Circles AY"Z" and BZ"X" have similar equations that may be written down by cyclic change of x, y, z and a, b, c. These circles meet at the point D" with co-ordinates (x, y, z), where

$$\begin{aligned} x &= a^2(a^2 - b^2)(c^2 - a^2), \\ y &= b^2(b^2 - c^2)(a^2 - b^2), \\ z &= c^2(c^2 - a^2)(b^2 - c^2). \end{aligned} \quad (10.2)$$

It may now be checked that this point lies on the circumcircle of ABC.

The equation of circle AY"C is

$$c^2a^2y^2 + a^2(b^2 - a^2)yz - b^2(c^2 + a^2 - b^2)zx + c^2(b^2 - c^2)\, xy = 0. \qquad (10.3)$$

Circles BZ"A and CX"B have similar equations to Equation (10.3) and may be written down by cyclic change of x, y, z and a, b, c.

It may now be checked that all these circles pass through the circumcentre O.

## 11. Another case with a similar configuration

It may be asked whether there are other examples of pairs of three lines whose intersections have such interesting properties. And the answer to this question is that the author knows of one other case. This is when one has the three altitudes AH, BH, CH and the three radii of the circumcircle AO, BO, CO. Points are now defined with X'= BH^CO, Y' = CH^AO, Z' = AH^BO, X = CH^BO, Y = AH^CO, Z = BH^AO. It then follows that circles XYZ, XY'Z', YZ'X', ZX'Y' all pass through the Brocard point Ω and the circles X'Y'Z', X'YZ, Y'ZX, Z'XY all pass through the Brocard point Ω'. It is also the case that circles XYZ and X'Y'Z' meet at H and again at a point J of the 7-point circle. See Figure 3.

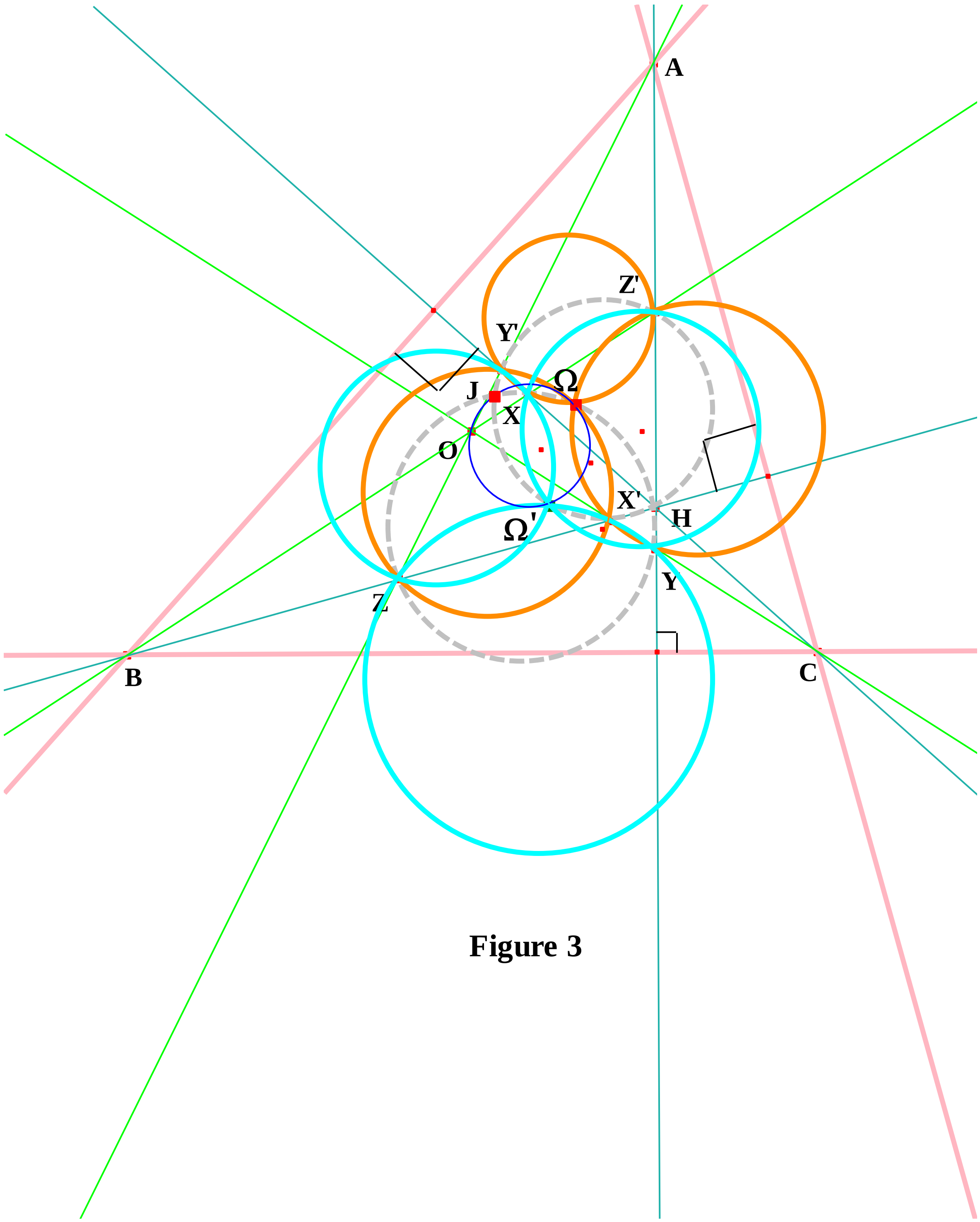



**Figure 3**

Proofs of all these results appear in Article 45 of Bradley [2].

## Appendix

**Some Observations on Pairs of Circles ABC and PQR having an intertwining property**

We say that circles ABC and PQR *intertwine* if circles ABR, BCP, CAQ pass through the same point S.

**Result 1 The intertwining property is symmetric**

Thus, if circles BCP, CAQ, ABR have a common point S, then circles QRA, RPB, PQC have a common point D.

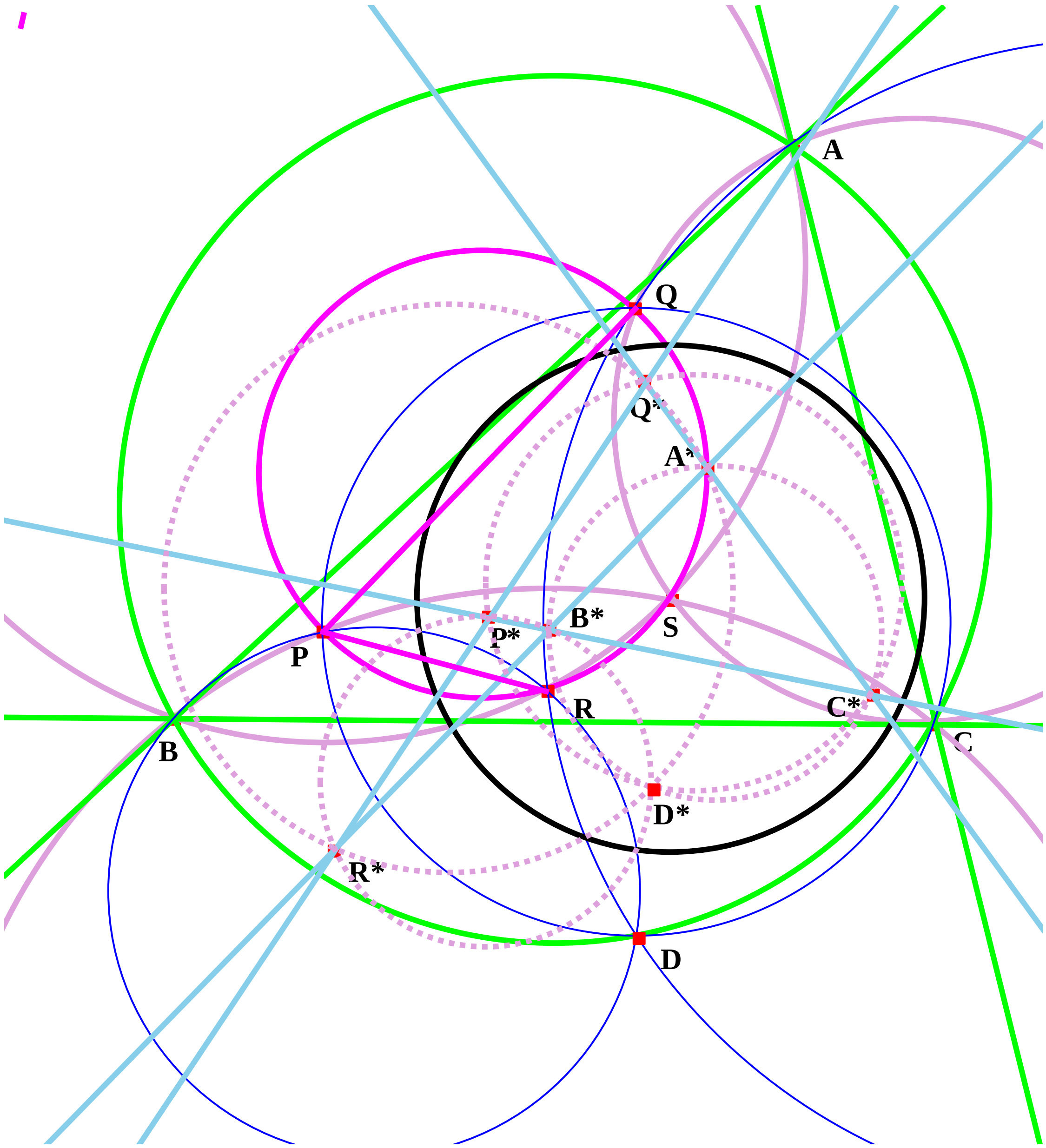
*Proof*
If S is the point of concurrence of circles BCP, CAQ and ABR, then invert with respect to S and the configuration becomes one in which (using stars for inverted points) P* lies on B*C*, Q* lies on C*A* and R* lies on A*B*. It then follows that circles Q*R*A*, R*P*B*, P*Q*C* are concurrent at the point D*, the Miquel point. The inverse image D of the point D* is now the point of concurrence of circles QRA, RPB and PQC.

**Result 2 If in Result 1 the point S lies on circle PQR, then the point D lies on circle ABC**

*Proof*
See Result 1. If S now lies on circle PQR, after inverting with respect to S, the point D* is the point of intersection of the four circumcircles of the four triangles formed by the lines P*Q*R*, P*B*C*, Q*C*A*, R*A*B*. The inverse image D of the point D* now lies on circle ABC. See the Figure above.

**Result 3 The intertwining property is transitive**

Thus if ABC and PQR intertwine and ABC and QRP also intertwine, then ABC and RPQ also intertwine. This result is not required in this article, but is added for the sake of completeness.
This time, after inversion, the result is still not immediate, and the details are left to the reader. A proof appears in Article 19, by the Author [2] on the Internet, entitled ‘Circular Perspective’.




*Acknowledgement*
I am grateful to David Monk [3] for pointing out the extension of Result 1 above, recorded as Result 2 and also for many useful exchanges about Section 11.

*References*


1. F.E. Wood, *Amer. Math. Monthly* 36:2 (1929) 67-73.
2. C.J.Bradley, Article 19 ,**http://people.bath.ac.uk/masgcs/bradley.html**
3. D. Monk, *private communication*.



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