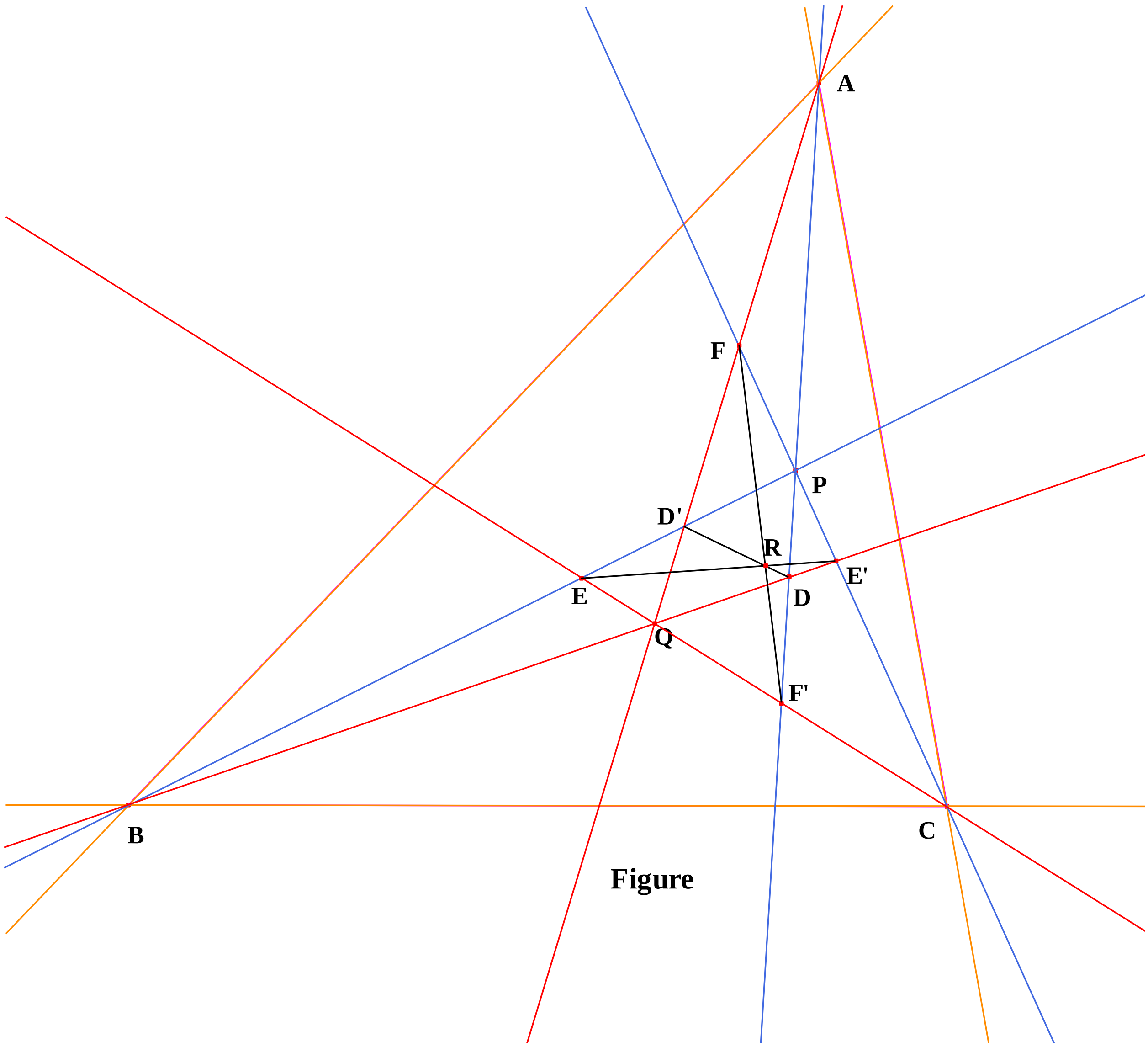
A Mean of two Cevian Points and the Construction of Triangle Centres

Christopher J Bradley



Figure

1. Introduction

The idea of *a triangle centre* is well known. Over 3000 such points have been investigated and tabulated [1]. There are, of course, an infinity of triangle centres because every point on a line joining any two of them is also a triangle centre. For example, the Euler line collapses to the centroid when a triangle becomes equilateral. So the nine-point centre is a triangle centre, and one knows this without the need to work out its co-ordinates. Obviously there has to be some additional reason as to what actually qualifies as a triangle centre to make it worth tabulating. Presumably it must involve a construction of some kind relating it to the triangle or to other triangle centres that makes it significant or acceptable as an additional triangle centre. The aim of this short article is to show that this is a very difficult task. My understanding is that Clark Kimberling decides when a point is significant. And the reason is not presumably that it lies on a line joining any two of them! What we do is to choose two points, produce a significant construction to provide a third point, which is a triangle centre if the two originally chosen ones are. Does the third point qualify or not? If the two chosen points are the centroid and the orthocentre the third point turns out to be the symmedian point. Does this convince one that the third point should qualify? If it does, remember there are already about 3000 triangle centres, so this would only add about 4.5 million.

2. The construction

We choose any two points P and Q, not on the sides of the triangle and not so as to be on the same line through a vertex. Let P have co-ordinates (p, q, r) and Q have co-ordinates (u, v, w). AP has equation qz = ry and BQ has equation wx = uz. These lines meet at the point X(ur, wq, wr). BP^CQ = Y(up, vp, ru) and CP^AQ = Z(vp, vq, wq). Next we interchange P and Q and form X' = AQ^BP with co-ordinates (wp, vr, wr), BQ^CP = Y'(up, uq, wp), CQ^AP = Z'(uq, vq, vr).

The equation of XX' may now be calculated and is found to be

$$wr(wq - vr)x + wr(wp - ur)y + (r^2uv - w^2pq)z = 0. \qquad (2.1)$$

The equation of YY' may be obtained from this by cyclic change of x, y, z and p, q, r and u, v, w.

We now obtain the point of intersection R with co-ordinates

$$x = up(wq + vr),\quad y = vq(wp + ur),\quad z = wr(vp + uq). \qquad (2.2)$$

See the figure above. It is easy to see that if v, w › u and q, r › p , so that P, Q are triangle centres, then R is also a triangle centre. In particular if P is the centroid then R has co-ordinates (u(v + w), v(w + u), w(u + v)), and further if Q is the orthocentre, then R is the symmedian point.

Reference

1. Clark Kimberling, "*Major Centers* of *Triangles*," Amer. Math. Monthly 104 (1997) 431-438.

Flat 4
Terrill Court
12-14, Apsley Road,
BRISTOL BS8 2SP