

ARTICLE 4

STUDIES IN SIMILARITY

Christopher J Bradley

1. Introduction

Given a scalene triangle ABC there are precisely six points J such that the angles BJC , CJA , AJB have values $\alpha = 180^\circ - A$, $\beta = 180^\circ - B$, $\gamma = 180^\circ - C$ in some order. Three of these points are well known to be the orthocentre H and the two Brocard points Ω and Ω' , which in this article, for reasons that will almost immediately become clear, we denote by $H+$ and $H-$. The other three points are less familiar. We denote them by aH , bH , cH and they lie on the circumference of the orthocentroidal circle (diameter GH), where the medians meet this circle (other than at the centroid G).

The angles BJC , CJA , AJB at these points are shown in the Table 1 below.

J	$\angle BJC$	$\angle CJA$	$\angle AJB$
H	α	β	γ
$H+$	γ	α	β
$H-$	β	γ	α
aH	α	γ	β
bH	γ	β	α
cH	β	α	γ

The notation used for the six points is prompted by Table 1, which reflects the group of symmetries of an equilateral triangle.

Each Brocard point may be constructed by drawing three circles and these circles together with the circles BHC , CHA , AHB produce nine circles. We first show how these nine circles may be used to locate the points aH , bH , cH . We determine the equations of the nine circles and the coordinates of aH , bH , cH showing that they have the locations stated above. We also find the coordinates of the centres of these nine circles. The nine centres have the following remarkable properties. They form six triangles with each centre appearing as a vertex of two of the triangles. Each of these triangles is similar to triangle ABC , three of them being directly similar and three being indirectly similar (with vertices in the opposite order). Also the three that are indirectly similar are mutually in perspective with O , the circumcentre of ABC , as perspector.

Given a triangle ABC with orthocentre H then, if we draw a general circle through H , its points of intersection X , Y , Z with the altitudes AH , BH , CH respectively, determine a triangle XYZ that is indirectly similar to triangle ABC . The circles through H are known as Hagge circles [1]. Since

the similarity is indirect there is a centre of inverse symmetry P and lines through P , which are the axes of inverse symmetry. If AP, BP, CP meet the circumcircle again at D, E, F and XP, YP, ZP meet the Hagge circle again at U, V, W then triangles DEF and UVW are also similar. There is an obvious connection between the vertices of the two triangles, which is that U is the reflection of D in BC , V the reflection of E in CA and W the reflection of F in AB .

The reason a Hagge circle carries a triangle that is indirectly similar to ABC by means of the construction mentioned above is that the angles subtended at H by the sides BC, CA, AB are α, β, γ respectively. The proof of this is a straightforward argument using angle chasing.

One might therefore hope, that if we use either of the Brocard points $H+$ or $H-$ and draw a general circle through that Brocard point and find its intersections with the line segments from that Brocard point to the vertices, then these intersections might determine a triangle ZXY or YZX , which is indirectly similar to triangle ABC . This indeed proves to be the case and again there is a centre and axis of inverse symmetry, see Bradley [2].

If one follows the same procedure with any of the three points aH, bH, cH we get triangles that turn out to be directly similar to triangle ABC . Direct similarities have rather different properties from indirect similarities and these properties are the subject of a comprehensive survey by Wood [3]. We point out how our new circles and triangles fit into Wood's scheme.

2. The nine circles and the six triangles

We first consider three circles that determine H . If you reflect O in the side BC to obtain the point aA and then draw a circle centre aA of radius R (the circumradius of ABC), then this circle is the circle BHC . It is also the case that, if bB and cC are the reflections of O in CA and AB respectively, then circles centres bB and cC of radius R are respectively the circles CHA and AHB . The common point of these three circles is therefore the orthocentre H . In terms of areal co-ordinates, since the co-ordinates of H are given by $(1/(b^2 + c^2 - a^2), 1/(c^2 + a^2 - b^2), 1/(a^2 + b^2 - c^2))$, the equations of these circles are

$$BHC: \quad a^2yz + b^2zx + c^2xy - (b^2 + c^2 - a^2)x(x + y + z) = 0, \quad (2.1)$$

$$CHA \quad a^2yz + b^2zx + c^2xy - (c^2 + a^2 - b^2)y(x + y + z) = 0, \quad (2.2)$$

$$AHB \quad a^2yz + b^2zx + c^2xy - (a^2 + b^2 - c^2)z(x + y + z) = 0. \quad (2.3)$$

Using the fact that the centre of a circle is the pole of the line at infinity we find the unnormalised co-ordinates of the centres are

$$aA(-a^2(b^2 + c^2 - a^2), -a^4 - c^4 + a^2b^2 + 2c^2a^2 + b^2c^2, -a^4 - b^4 + 2a^2b^2 + c^2a^2 + b^2c^2), \quad (2.4)$$

$$bB(-b^4 - c^4 + 2b^2c^2 + a^2b^2 + c^2a^2, -b^2(c^2 + a^2 - b^2), -b^4 - a^4 + b^2c^2 + 2a^2b^2 + c^2a^2), \quad (2.5)$$

$$cC(-c^4 - b^4 + c^2a^2 + 2b^2c^2 + a^2b^2, -c^4 - a^4 + 2c^2a^2 + b^2c^2 + a^2b^2, -c^2(a^2 + b^2 - c^2)). \quad (2.6)$$

Triangle $aAbBcC$ is congruent to triangle ABC being the image of ABC under an 180° rotation about the nine-point centre N .

We next consider the three circles defining the Brocard point $H+$, which has co-ordinates $(1/b^2, 1/c^2, 1/a^2)$. These are well known to be (i) the circle through A and B touching BC at B , (ii) the

circle through B and C and touching CA at C and (iii) the circle through C and A touching AB at A . We denote their centres by aB , bC , cA respectively. The equations of these circles are:

$$AH+B: \quad a^2z^2 + (a^2 - b^2)zx - c^2xy = 0, \quad (2.7)$$

$$BH+C: \quad b^2x^2 + (b^2 - c^2)xy - a^2yz = 0, \quad (2.8)$$

$$CH+A: \quad c^2y^2 + (c^2 - a^2)yz - b^2zx = 0. \quad (2.9)$$

The unnormalised co-ordinates of their centres are

$$aB(2c^2a^2, -a^4 - b^4 + 2a^2b^2 + c^2a^2 + b^2c^2, -c^2(c^2 + a^2 - b^2)), \quad (2.10)$$

$$bC(-a^2(a^2 + b^2 - c^2), 2a^2b^2, -b^4 - c^4 + 2b^2c^2 + a^2b^2 + c^2a^2), \quad (2.11)$$

$$cA(-c^4 - a^4 + 2c^2a^2 + b^2c^2 + a^2b^2, -b^2(b^2 + c^2 - a^2), 2b^2c^2). \quad (2.12)$$

We next consider the three circles defining the Brocard point H -, which has co-ordinates $(1/c^2, 1/a^2, 1/b^2)$. These are well known to be (i) the circle through A and C touching BC at C , (ii) the circle through B and A and touching CA at A and (iii) the circle through C and B touching AB at B . We denote their centres by aC , bA , cB respectively. The equations of these circles are:

$$CH-A: \quad a^2y^2 - b^2zx + (a^2 - c^2)xy = 0, \quad (2.13)$$

$$AH-B: \quad b^2z^2 - c^2xy + (b^2 - a^2)yz = 0, \quad (2.14)$$

$$BH-C: \quad c^2x^2 - a^2yz + (c^2 - b^2)zx = 0. \quad (2.15)$$

The unnormalised co-ordinates of their centres are

$$aC(2a^2b^2, -b^2(a^2 + b^2 - c^2), -c^4 - a^4 + a^2b^2 + 2c^2a^2 + b^2c^2), \quad (2.16)$$

$$bA(-a^4 - b^4 + b^2c^2 + 2a^2b^2 + c^2a^2, 2b^2c^2, -c^2(b^2 + c^2 - a^2)), \quad (2.17)$$

$$cB(-a^2(c^2 + a^2 - b^2), -b^4 - c^4 + c^2a^2 + 2b^2c^2 + a^2b^2, 2c^2a^2). \quad (2.18)$$

The co-ordinates of the nine centres are unnormalised but in each case they sum to the same amount $(a + b + c)(b + c - a)(c + a - b)(a + b - c)$. The nine circles are illustrated in Fig. 1. Also shown in Fig. 1 is the orthocentroidal circle on GH as diameter, with equation shown in Bradley and Smith [4] to be

$$(b^2 + c^2 - a^2)x^2 + (c^2 + a^2 - b^2)y^2 + (a^2 + b^2 - c^2)z^2 - a^2yz - b^2zx - c^2xy = 0. \quad (2.19)$$

The median AG meets this circle again at the point aH with co-ordinates

$aH(a^2, b^2 + c^2 - a^2, b^2 + c^2 - a^2)$. To prove that this is indeed the point aH featured in Section 1, one may verify that it lies on each of the circles with centres aA , aB , aC and consequently the sides BC , CA , AB subtend at aH the angles α , γ , β these being the angles in the segments of those circles. Similarly bH and cH lie on the orthocentroidal circle and on the medians BG and CG respectively and have co-ordinates $bH(c^2 + a^2 - b^2, b^2, c^2 + a^2 - b^2)$ and $cH(a^2 + b^2 - c^2, a^2 + b^2 - c^2, c^2)$. The point bH lies on the circles with centres bA , bB , bC and cH lies on the circles with centres cA , cB , cC .

3. Lengths of segments and the similarities

If ABC is an acute-angled triangle the distances from the vertices to the orthocentre and to the Brocard points are known, see Shail [5], to be

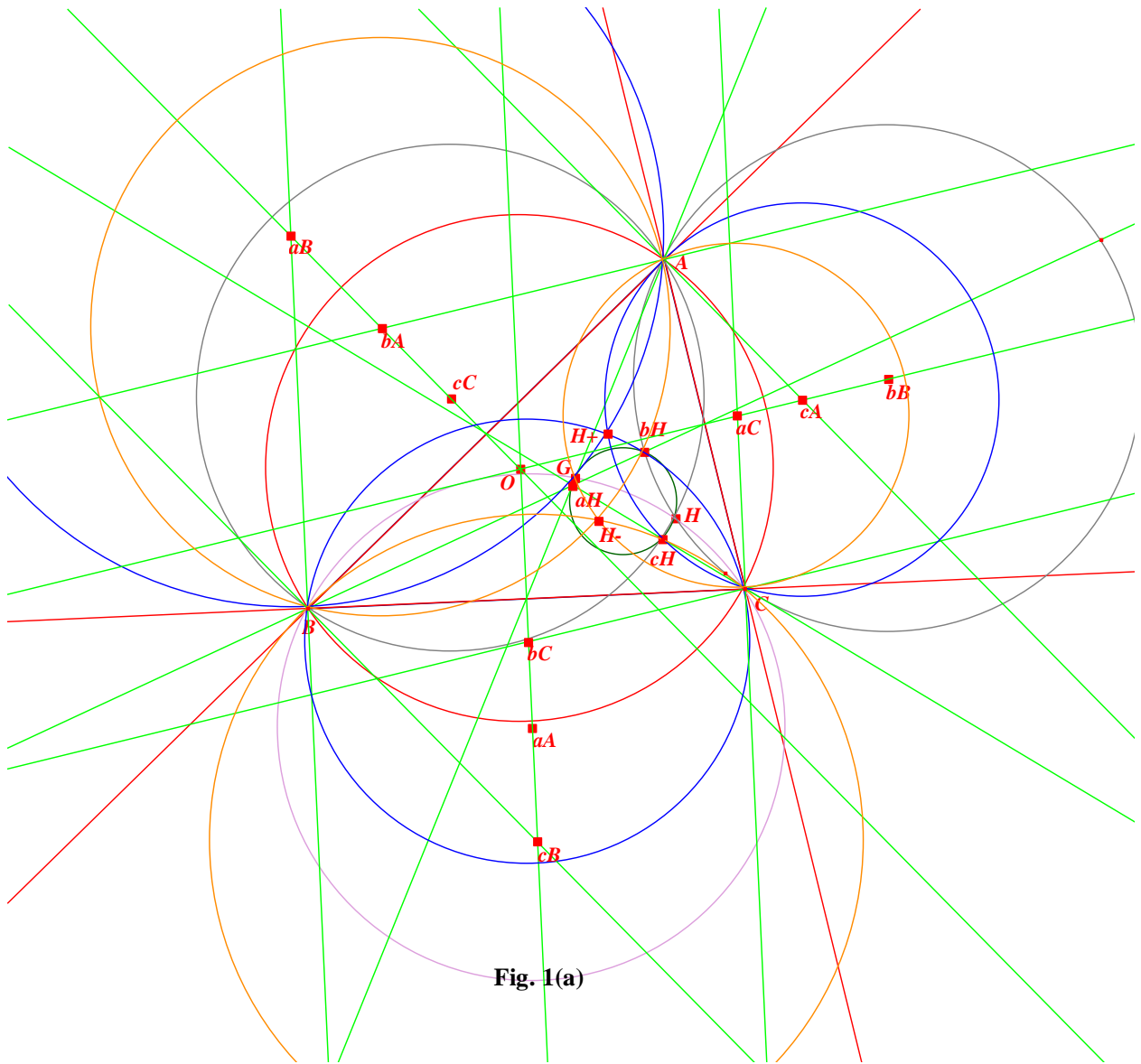
$$AH = 2R \cos A, \quad BH = 2R \cos B, \quad CH = 2R \cos C;$$

$$AH+ = 2R_0 \sin B/\sin A, \quad BH+ = 2R_0 \sin C/\sin B, \quad CH+ = 2R_0 \sin A/\sin C;$$

$AH = 2R_0 \sin C / \sin A$, $BH = 2R_0 \sin A / \sin B$, $CH = 2R_0 \sin B / \sin C$,
 where $R_0 = \frac{1}{2} abc / (b^2 c^2 + c^2 a^2 + a^2 b^2)^{1/2}$. The reader may verify that the corresponding formulae
 for aH , bH , cH are

$A aH = 4R_0 \cos A$, $B aH = 2R_0 \sin A / \sin B$, $C aH = 2R_0 \sin A / \sin C$,
 $A bH = 2R_0 \sin B / \sin A$, $B bH = 4R_0 \cos B$, $C bH = 2R_0 \sin B / \sin C$,
 $A cH = 2R_0 \sin C / \sin A$, $B cH = 2R_0 \sin C / \sin B$, $C cH = 4R_0 \cos C$,

where R_0 is, in each case, the radius of the pedal triangle of the corresponding point. In an
 obtuse-angled triangle these become signed lengths. For the point aH the value of
 $R_0 = \frac{1}{2} bc / (2b^2 + 2c^2 - a^2)^{1/2}$ with a similar formula for the points bH , cH .



We next consider the triangle $cA aB bC$ and show that it is similar to triangle ABC .

The displacement $bC aB = (a^2(a^2 + b^2 + c^2), -a^4 - b^4 + c^2a^2 + b^2c^2, b^4 - b^2c^2 - a^2b^2 + 2c^2a^2$

Using the areal metric, see Bradley [6], we find that $(bC aB)^2 = ka^2$, where

$$k = (3a^2b^2c^2(a^2 + b^2 + c^2) + 2(b^4c^4 + c^4a^4 + a^4b^4) - a^6(b^2 + c^2) - b^6(c^2 + a^2) - c^6(a^2 + b^2)) \quad (3.1)$$

all divided by $\{(a + b + c)(b + c - a)(c + a - b)(a + b - c)\}^2$. Similarly $(cA bC)^2 = kb^2$ and $(aB cA)^2 = kc^2$, from which the result follows. Another short calculation shows that triangle $bA cB aC$ is congruent to triangle $cA aB bC$. These two triangles together with ABC and $aA bB cC$ are all directly similar to one another and fall into two congruent pairs.

We now consider the three triangles $aA aB aC$, $bA bB bC$ and $cA cB cC$. Note first that the centres aA , bC , cB all lie on the perpendicular bisector of the side BC . Similarly the centres bB , cA , aC all lie on the perpendicular bisector of CA and the centres cC , aB , bA all lie on the perpendicular bisector of AB . It follows that triangles $aA aB aC$, $bC bA bB$, $cB cC cA$ are mutually in perspective with the common perspector O .

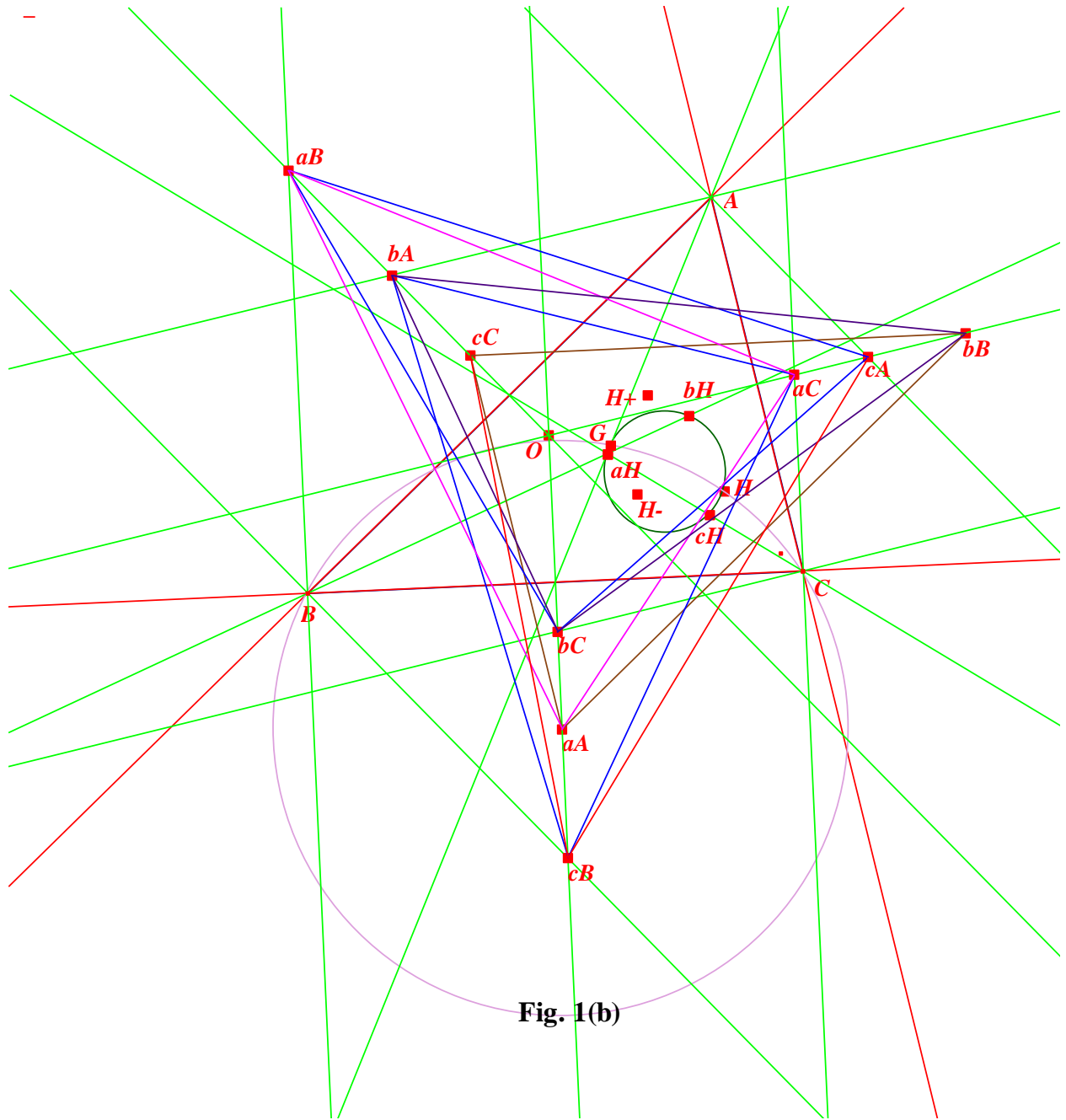


Fig. 1(b)

The displacement $\mathbf{aB aC} = (2a^2(b^2 - c^2), a^2(a^2 - 3b^2 - c^2), a^2(b^2 + 3c^2 - a^2))$. Using the areal metric we find $(\mathbf{aB aC})^2 = la^2$, where

$$l = \{a^2(2b^2 + 2c^2 - a^2)\} / \{(a + b + c)(b + c - a)(c + a - b)(a + b - c)\}. \quad (3.2)$$

Similar calculations show that $(\mathbf{aC aA})^2 = lb^2$ and $(\mathbf{aA aB})^2 = lc^2$. It follows that triangle $\mathbf{aA aB aC}$ is similar to triangle ABC .

In similar manner we find $(\mathbf{bB bC})^2 = ma^2$, $(\mathbf{bC bA})^2 = mb^2$ and $(\mathbf{bA bB})^2 = mc^2$, where

$$m = \{b^2(2c^2 + 2a^2 - b^2)\} / \{(a + b + c)(b + c - a)(c + a - b)(a + b - c)\}, \quad (3.3)$$

showing that triangle $\mathbf{bA bB bC}$ is also similar to triangle ABC .

And again in similar manner we find $(\mathbf{cB cC})^2 = na^2$, $(\mathbf{bC bA})^2 = nb^2$ and $(\mathbf{bA bB})^2 = nc^2$, where

$$n = \{c^2(2a^2 + 2b^2 - c^2)\} / \{(a + b + c)(b + c - a)(c + a - b)(a + b - c)\}, \quad (3.4)$$

showing that triangle $bA bB bC$ is also similar to triangle ABC .

All three triangles are directly similar to one another and inversely similar to triangle ABC . The areas of the three triangles are in the ratio $l : m : n$. The six triangles are shown in Fig.1(b).

References

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Flat 4,
Terrill Court,
12-14 Apsley Road,
BRISTOL BS8 2SP.