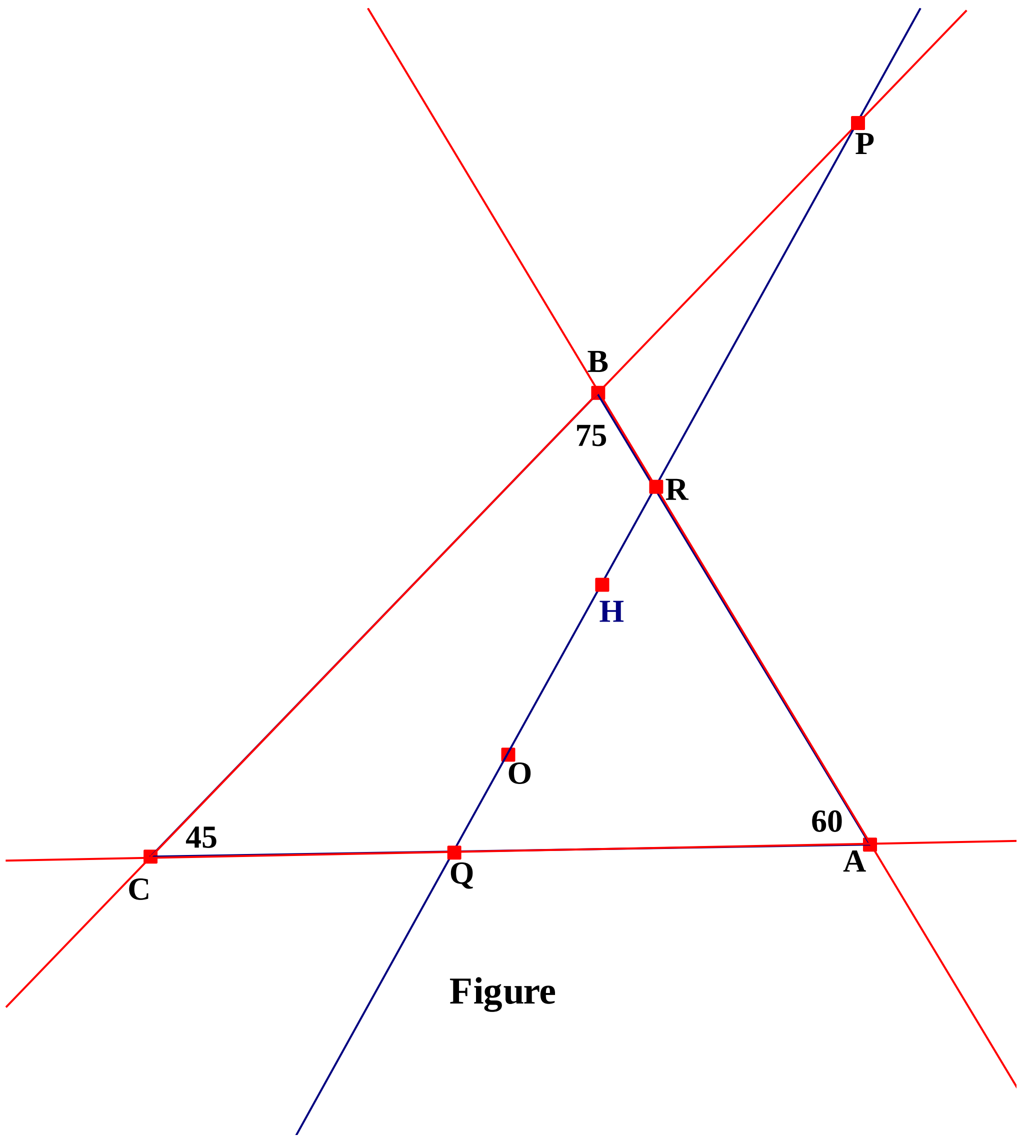
Article 39
The 60^o, 75^o, 45^o Triangle and its Euler line

Christopher J Bradley



Figure

1. Introduction

Let ABC be a 60^o, 75^o, 45^o triangle and suppose that the Euler line OH, where O is the circumcentre and H the orthocentre, meets the sides BC, CA, AB at P, Q, R respectively. We prove that PR = RQ and RH = OQ. It is also the case that AQ = AR so that angle BPR = 15^o.

2. The sides of the triangle and u, v, w

The sides are proportional to the sines of the angles of the triangle and may therefore be taken as $a = \frac{1}{2}\sqrt{3}$, $b = \frac{1}{4}(\sqrt{6} + \sqrt{2})$, $c = \frac{1}{2}\sqrt{2}$. In the sections that follow we use the co-ordinates of H(u, v, w), where

$$u = 1/(b^2 + c^2 - a^2),\ v = 1/(c^2 + a^2 - b^2),\ w = 1/(a^2 + b^2 - c^2). \qquad (2.1)$$

3. The Euler line and the points P, Q, R

The Euler line passes through H(u, v, w) and O(v + w, w + u, u + v) and so has the equation

$$(v - w)x + (w - u)y + (u - v)z = 0. \qquad (3.1)$$

The Euler line meets BC, x = 0, at P(0, 1 + (1/3)√3, – (1/3)√3). It meets CA, y = 0 at Q(1 – (1/3)√3, 0, (1/3)√3), and it meets AB, z = 0 at R(½ – (1/6)√3, (1/6)√3 + ½, 0). It is clear that PR = RQ, since $x_P + x_Q = 2x_R$.

4. O, H and RH = OQ

The values of u, v, w given in Equation (2.1), which when normalized, are the co-ordinates of H(2 – √3, (1/3)√3, (2/3)√3 – 1) and those of O are (½√3 – ½ , ½ –(1/6)√3, 1 – (1/3)√3).

It now follows that $x_H + x_O = x_Q + x_R$ so that RH = OQ.

From the co-ordinates of Q and R and the side lengths b and c it immediately follows that triangle RQA is equilateral and hence that angle BPR = 15^o.

5. The circumcircle of ABC, Circle CQH and the point U

The equation of the circumcircle, using u, v, w, is

$$u(v + w)yz + v(w + u)zx + w(u + v)xy = 0. \qquad (5.1)$$

This becomes

$$(9\sqrt{3} - 15)yz + (\sqrt{3} - 1)zx + (6\sqrt{3} - 10)xy = 0. \qquad (5.2)$$

The equation of the circle CQH is

$$(\sqrt{3} - 1)x^2 + (33 - 19\sqrt{3})y^2 + 2(3 - 2\sqrt{3})yz + 2(\sqrt{3} - 2)zx + 2(7 - 4\sqrt{3})xy = 0. \qquad (5.3)$$

These two circles meet at C and again at a point U with co-ordinates

$$(1/13)(7 - 5\sqrt{3}),\ (11//3)\sqrt{3} + 7,\ (4/3)\sqrt{3} - 1).$$

The point U is a key point in this configuration.

6. The circles BRH, COQ and the points S and T

The equation of the circle BRH is

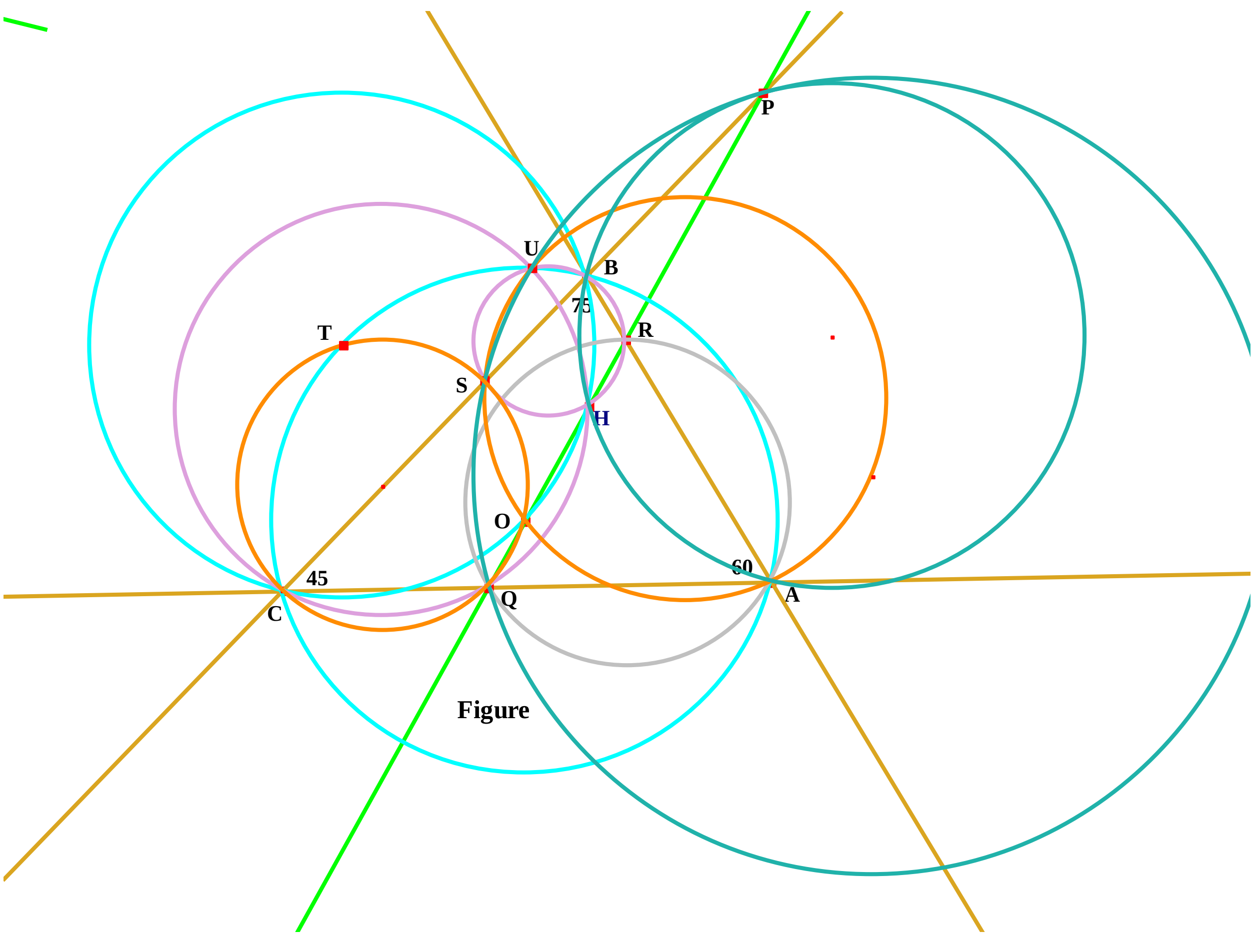
$$2(\sqrt{3} - 2)x^2 + 2(5\sqrt{3} - 9)z^2 + (9 - 5\sqrt{3})yz + (11\sqrt{3} - 19)zx + 2(7 - 4\sqrt{3})xy = 0. \qquad (6.1)$$

It may now be verified that this circle passes through U. Also it meets BC at B and the point S with co-ordinates S(0, 2/3, 1/3). That is, where SC = 2BS. This is a pleasing and unexpected result.

The circle COQ has equation

$$(\sqrt{3} - 1)x^2 + (9 - 5\sqrt{3})y^2 + 2(5\sqrt{3} - 9)yz + 2(\sqrt{3} - 2)zx + 2(3\sqrt{3} - 5)xy = 0. \qquad (6.2)$$

It may now be verified that this circle passes through the point S. It also meets the circumcircle at a point T with co-ordinates (1/6)(3 – 3√3, √3 + 3, 2√3). The points T, S, H turn out to be collinear.



Figure

7. Circles AOU and PUQ pass through S

The equation of circle AOU is

$$2(7 - 4\sqrt{3})y^2 - 2(\sqrt{3} - 1)z^2 + (17\sqrt{3} - 29)yz + (\sqrt{3} - 1)zx + 2(5\sqrt{3} - 8)xy = 0. \qquad (7.1)$$

It may now be verified that this circle passes through S.

The equation of circle PUQ is

$$(\sqrt{3} - 1)x^2 + (3\sqrt{3} - 5)y^2 + 4(\sqrt{3} - 2)z^2 + 2(7 - 4\sqrt{3})yz + 2(2\sqrt{3} - 3)zx + 6(2 - \sqrt{3})xy = 0. \qquad (7.2)$$

And this circle also passes through S.

8. Circle ABH passes through P

The equation of circle ABH is

$$2(3 - 2\sqrt{3})z^2 + (5\sqrt{3} - 9)yz + (5 - 3\sqrt{3})zx + 2(3\sqrt{3} - 5)xy = 0. \qquad (8.1)$$

It may now be verified that this circle passes through P.

Finally it may now be verified that circles ABH and PUQ touch at P.

Flat 4
Terrill Court
12-14 Apsley Road
BRISTOL BS8 2SP