Article 36
Where 7 Circles meet Part 2

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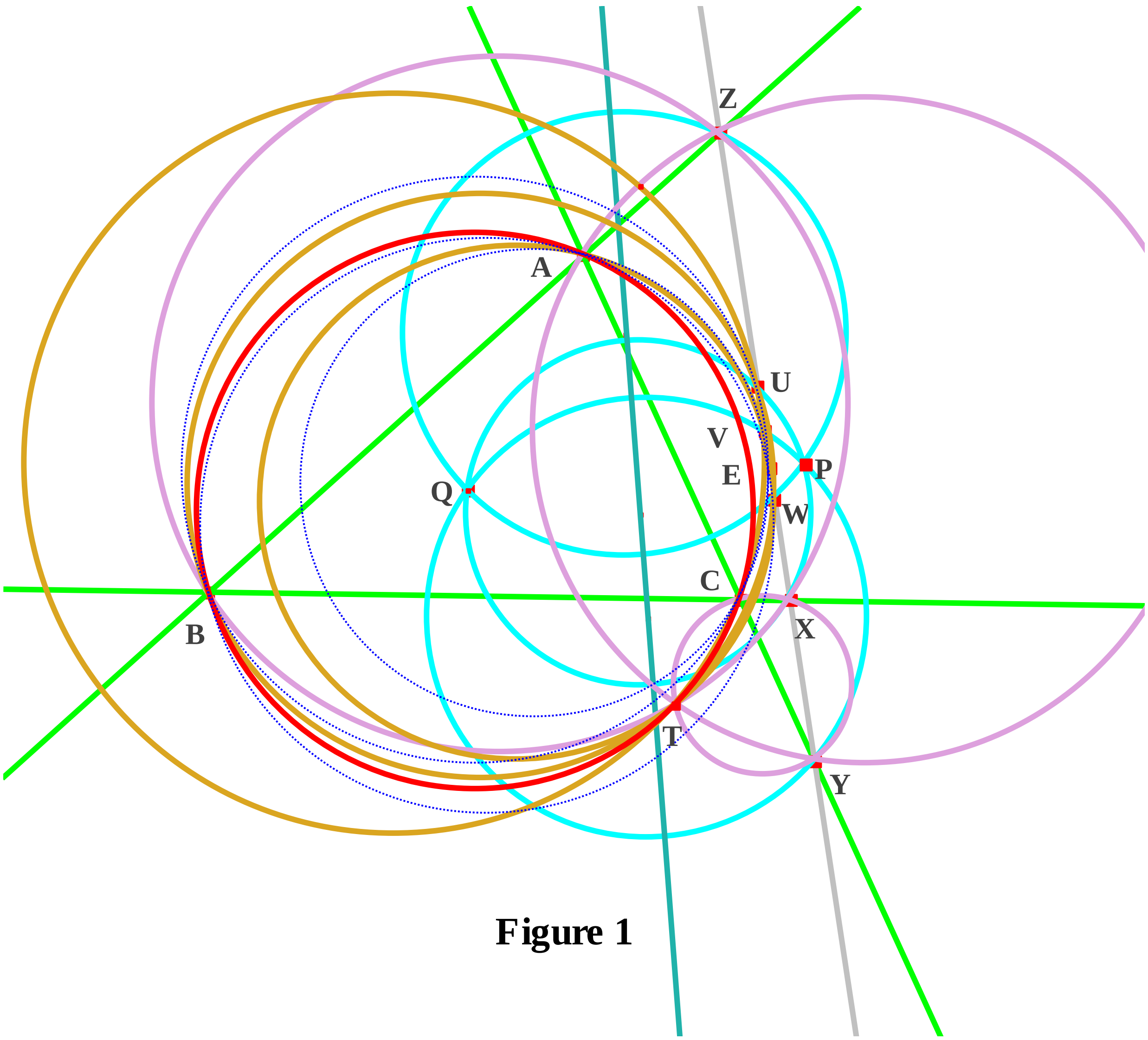

Figure 1

1. Introduction

This article should be read as the last of a sequence of articles, starting with Article 19 and continuing through 30 – 33 and 35, on circular perspective. It deals with case when a triangle

ABC is in circular perspective with a pair of triangles both of which have vertices on a circle. What happens is that if a circle Σ is drawn, other than the circumcircle S and not through any vertex, and arbitrary points D and E are chosen on it, then if circles BCD, CAD, ABD are drawn to meet Σ again at X, Y, Z respectively and if circles BCE, CAE, ABE are drawn to meet Σ at U, V, W respectively, then the following happens: circles AYZ, BZX, CXY concur at a point T on S and circles AVW, BWU, CUV meet at a point T' on S. However, if E is chosen in a particular way then T and T' coincide. In fact for each choice of D on Σ, there is only one point E on Σ with the required property and when E is chosen appropriately, then UX, VY, WZ are concurrent.

What we do first is to consider a limiting case when the circle Σ degenerates into a line m. This situation is shown in Figure 1. The line m is a transversal of ABC and does not pass through any vertex. The points X, Y, Z are the points where m meets BC, CA, AB respectively. Q is a fixed point not at a vertex. It then follows that circles AYZ, BZX, CXY concur at the Miquel point T, which in the case of a transversal is well known to lie on the circumcircle S. The points U, V, W are now defined as the intersections of circles BCE, CAE, ABE with m. It now turns out that circles AVW, BWU, CUV always concur at a point T' on S. But there is one and only one position of E which results in the point T' coinciding with T. This turns out to be when the circles XUQ, YVQ, ZWQ are coaxal. Interestingly E does not move from its favourable position on m when the point Q is altered to another position.

The reader, by now, will no doubt have appreciated that the general problem is now proved by inverting Figure 1 through the point Q. The fixed point Q is relabelled D, the coaxal circles simply become the lines XU, YV, ZW. It is hoped that no confusion is caused by not putting stars on the points in the inverted figure.

The analysis for the configuration of Figure 1 is carried out using areal co-ordinates with ABC as triangle of reference and it is by no means straightforward. Some geometers might regard it as horrendous, but my retort would then be to ask them to prove it otherwise.

2. The Miquel point T for triangle ABC and the points X, Y, Z

We take the transversal to have equation

$$lx + my + nz = 0. \tag{2.1}$$

It follows that X, Y, Z have co-ordinates X(0, n, – m), Y(– n, 0, l), Z(m, – l, 0). The circle AYZ has equation

$$a^2yz + b^2zx + c^2xy + (x + y + z)(ux + vy + wz) = 0, \tag{2.2}$$

where $u = 0$, $v = c^2m/(l – m)$, $w= b^2n/(l – n)$. The line with equation $ux + vy + wz = 0$, with these values of u, v, w is the equation of the common chord of circle AYZ and the circumcircle S and therefore meets S at A and the point T with co-ordinates

$$x = a^2mn(l – m)(n – l),$$
$$y = b^2nl(l – m)(m – n), \quad (2.3)$$
$$z = c^2lm(m – n)(n – l).$$

The symmetry of these co-ordinates under cyclic change of x, y, z and a, b, c and l, m, n shows that T also lies on circles BZX, CXY and is therefore the Miquel point.

3. The points U, V, W when E is chosen at random on XYZ

In what follows it is best to parameterize points on the transversal XYZ, with Equation (2.1), with a parameter s such that a general point on it has co-ordinates $(smn, (1 – s)nl, – lm)$. We now choose E to have parameter t so that E has co-ordinates $(tmn, (1 – t)nl, – lm)$.

The circle EBC has an equation of the form of Equation (2.2) with $v = w = 0$ and

$$u = \{l(a^2l(t – 1) – t(b^2m + c^2n(t – 1)))\}/\{t(l(m + n(t – 1)) – mnt)\}. \quad (3.1)$$

This circle meets XYZ at points with parameter s, where $s = t$ (the point E) and at U with

$$s = \{a^2l(l(m + n(t – 1)) – mnt)\}/\{n(a^2l(l – m)(t – 1) – mt(b^2(l – m) + c^2(n – 1)))\}. \quad (3.2)$$

The circle EAB has an equation of the form of Equation (2.2), with $u = v = 0$ and

$$w = \{n(a^2l(t – 1) – t(b^2m + c^2n(t – 1)))\}/\{mnt – l(m + n(t – 1))\}. \quad (3.3)$$

This circle meets XYZ at points with parameter s, where $s = t$ (the point E) and at W with

$$s = \{l(a^2m(l – n) + (n – m)(b^2m + c^2n(t – 1)))\}/\{c^2n(l(m + n(t – 1)) – mnt)\}. \quad (3.4)$$

The circle ECA has an equation of the form of Equation (2.2), with $u = w = 0$ and

$$v = \{m(a^2l(t – 1) – t(b^2m + c^2n(t – 1)))\}/\{(1 – t)(l(m + n(t – 1)) – mnt)\}. \quad (3.5)$$

This circle meets XYZ at points with parameter s, where $s = t$(the point E) and at V with

$$s = \{l(a^2n(l – m)(t – 1) + (m – n)(b^2m + c^2n(t – 1)))\}/\{n(a^2l(l – m)(t – 1) + b^2mt(m – 1) + c^2l(m – n)(t – 1))\}. \quad (3.6)$$

The co-ordinates of U, V, W now follow from Equations (3.2), (3.4) and (3.6).

4. The circles AVW, BWU, CUV and the point T'

We omit calculation of the equations of the circles AVW, BWU, CUV which though arduous is straightforward. We record the co-ordinates of the point T' where they are concurrent on the circumcircle. Its co-ordinates are (x, y, z), where

$$x = -a^2/\{a^2l(l(m - n(t - 1)^2) + mnt(t - 2)) + mnt^2(b^2(l - m) + c^2(n - 1))\}, \quad (4.1)$$

$$y = -b^2/\{a^2ln(l - m)(t - 1)^2 + b^2m(mnt^2 - l(m + n)(t^2 - 1))) + c^2ln(m - n)(t - 1)^2\}, \quad (4.2)$$

$$z = c^2/\{a^2lm(l - n) + b^2lm(n - m) + c^2n(mnt^2 - l(m(2t - 1) + n(t - 1)^2))\}. \quad (4.3)$$

For the points T and T' to coincide the ratios of their co-ordinates must be equal. This turns out to be a quadratic equation in t with one root equal to l(n – m)/n(l – m). However this would put the point E at infinity, which is not of interest as U, V, W would then coincide with X, Y, Z. This leaves a unique value of t, which is

$$t = \{l(a^2(l(m + n) - 2mn) + (n - m)(b^2m - c^2n))\}/\{n(a^2l(l - m) + b^2m(m - l) + c^2(l(2m - n) - mn))\}. \quad (4.4)$$

5. The positions of E, U, V, W when T and T' coincide

The value of t in Equation (4.4) immediately gives us the co-ordinates of E when it assumes its special position. These are (x, y, z), where

$$x = a^2(l(m + n) - 2mn) + (n - m)(b^2m - c^2n),$$
$$y = a^2l(n - l) + b^2(l(m - 2n) + mn) + c^2n(l - n), \quad (5.1)$$
$$z = a^2l(m - l) + b^2m(l - m) + c^2(mn - l(2m - n)).$$

The circle EAB has an equation of the form (2.2) with u = v = 0 and

$$w = -\{a^4l^2 - 2a^2l(b^2m + c^2n) + b^4m^2 - 2b^2c^2mn + c^4n^2\}/\{2(a^2l(l - m) + b^2m(m - l) + c^2(l(2m - n) - mn))\}. \quad (5.2)$$

This circle meets XYZ at E and again at the special position of W with co-ordinates (x, y, z), where

$$x = m(a^2l - b^2m + c^2n),$$
$$y = -l(a^2l - b^2m - c^2n), \quad (5.3)$$
$$z = -2lmc^2.$$

We now take the point Q to have co-ordinates Q(p, q, r). The circle QWZ has an equation of the form (2.2), where u, v, w are given by the equations

$$a^2qr + b^2rp + c^2pq + (p + q + r)(pu + qv + rw) = 0, \quad (5.4)$$

$$(l - m)(lv - mu) - c^2lm = 0, \quad (5.5)$$

$$a^4(c^2l^3m + l^2(l - m)(lv - mu)) - 2a^2l(b^2m(c^2lm + (l - m)(lv - mu)) + c^2(c^2lmn + l^2(nv - m(v + w)) + lm^2(u + w) - m^2nu)) + b^4m^2(c^2lm + (l - m)(lv - mu)) - 2b^2c^2m(c^2lmn + l^2(m(v + u) - nv) - lm^2(u + w) + m^2nu) + c^6lmn^2 + c^4(l(2m - n) - mn)(l(2mw - nv) - mnu) = 0. \quad (5.6)$$

The circle EBC has the form of Equation (2.2) with v = w = 0 and

$$u = \{a^4l^2 - 2a^2l(b^2m + c^2n) + b^4m^2 - 2b^2c^2mn + c^4n^2\}/\{2(a^2(l(m + n) - 2mn) + (n - m)(b^2m - c^2n))\}. \quad (5.7)$$

This circle meets XYZ at E and again at the special position of U with co-ordinates (x, y, z), where

$$x = 2a^2mn, \quad y = -n(a^2l + b^2m - c^2n), \quad z = -m(a^2l - b^2m + c^2n). \quad (5.8)$$

The circle QUX has an equation of the form (2.2), where u, v, w are given by the equations

$$a^2qr + b^2rp + c^2pq + (p + q + r)(pu + qv + rw) = 0, \quad (5.9)$$

$$(m - n)(mw - nv) - a^2mn = 0, \quad (5.10)$$

$$a^6l^2mn - a^4(2b^2lm^2n + 2c^2lmn^2 + (2mn - l(m + n))(l(mw + nv) - 2mnu)) + a^2(b^2m - c^2n)(b^2m^2n - c^2mn^2 - 2(l(m^2w - n^2v) - mn(m(u + w) - n(u + v)))) + (m - n)(b^2m - c^2n)^2(mw - nv) = 0. \quad (5.11)$$

In the same way it can be shown that circle ECA meets XYZ at E and again at the special position of V with co-ordinates (x, y, z), where

$$x = -n(a^2l + b^2m - c^2n), \quad y = 2b^2nl, \quad z = -l(-a^2l + b^2m + c^2n). \quad (5.12)$$

Once the values of u, v, w have been obtained from Equations (5.4) – (5.6) and (5.9) – (5.11) the equation of the common chord of the circles QWZ and QUX may be obtained as

$$(ux + vy + wz)_{QWZ} = (ux + vy + wz)_{QUX} \quad (5.13)$$

In the same way the equation of the common chord of circles QWZ and QVY may be obtained and is found to coincide with the common chord having Equation (5.13). This proves that the condition for E to have special position on XYZ in order for T and T' to coincide the circles QUX, QVY, QWZ must be coaxal.

Inverting with respect to Q proves the more general theorem when the eight points D, E, X, Y, Z, U, V, W lie on a circle and the condition then for T and T' to coincide is that UX, VY, WZ are concurrent. See Figure 2 below

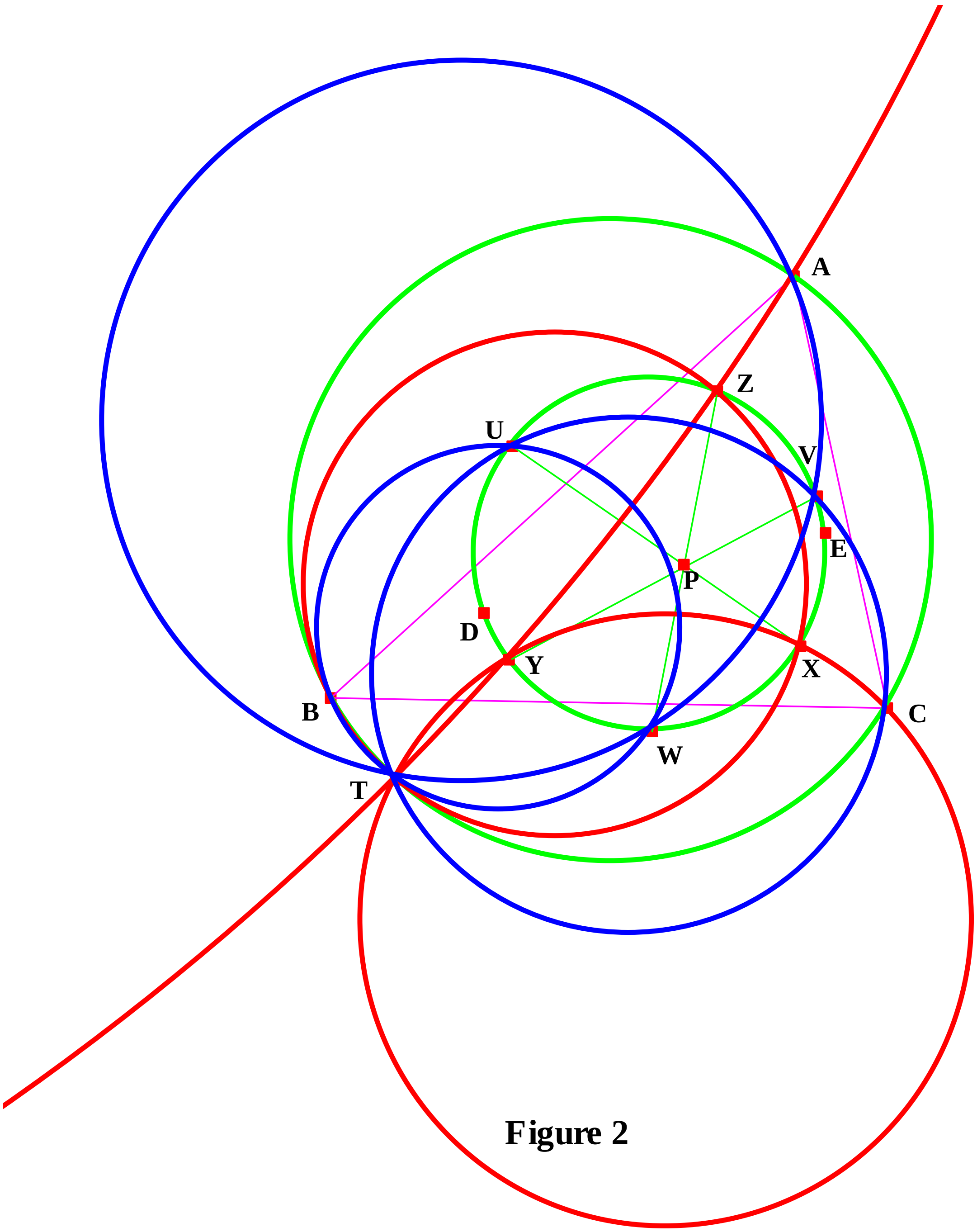

Figure 2

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