ARTICLE 32

The GH Disc and another case of Triple Circular Perspective

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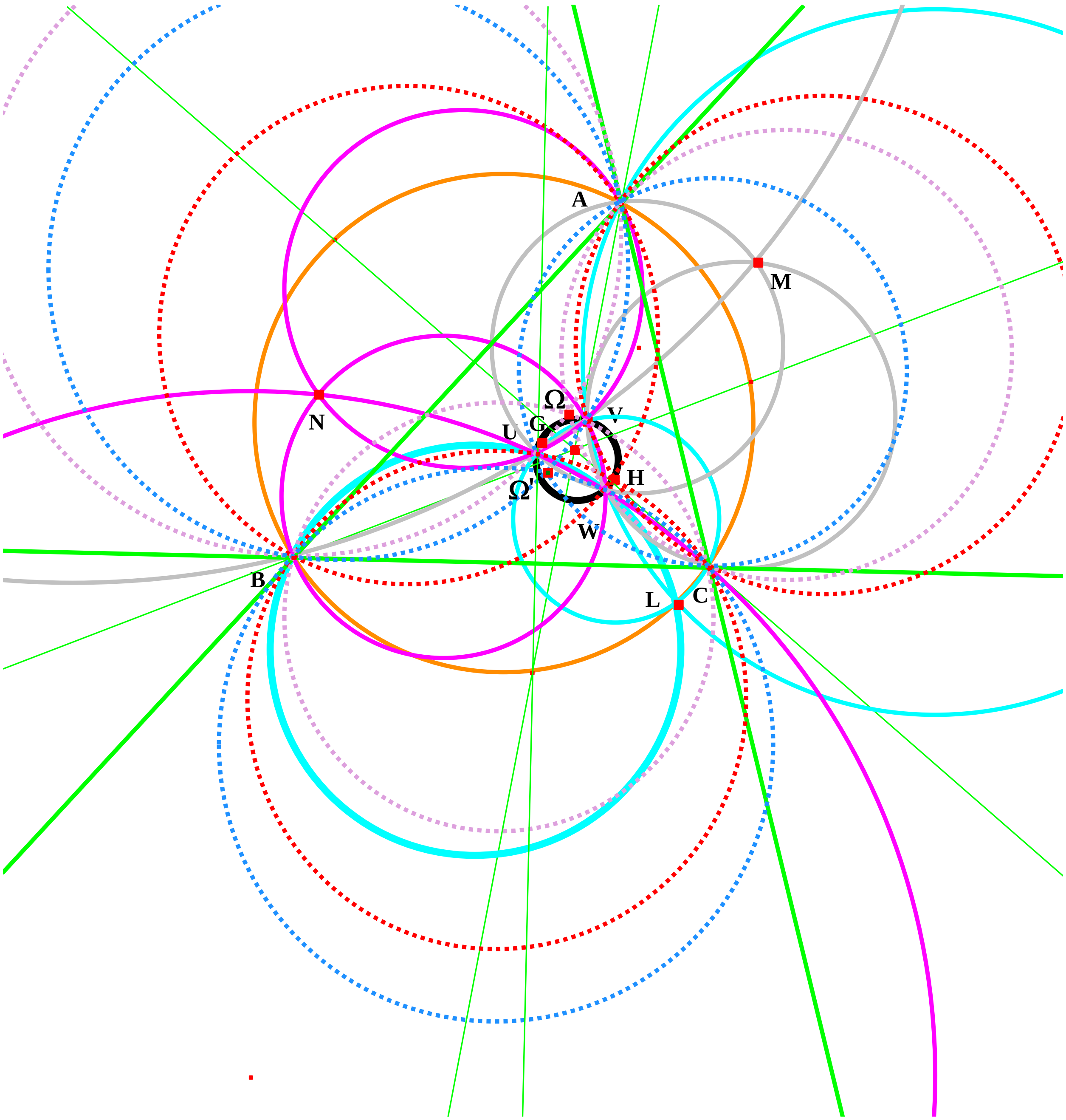

1. Introduction

This article should be read after Articles 19, 30, 31, in which circular perspectivity is defined and where particular emphasis is given to triple circular perspectivity and certain examples of this property. In this article we take three points U, V, W on the orthocentroidal circle S and show that triangle UVW is in triple circular perspective with ABC. Circles BUC, CVA, AWB meet at the orthocentre H, circles BVC, CWA, AUB meet at the first Brocard point Ω and circles BWC, CUA, AVB meet at the second Brocard point Ω'. Also circles AVW, BWU, CUV meet at a point L on the circumcircle of ABC, its counterpart H lying on the circumcircle of UVW (the orthocentroidal circle). In the above the point U may be defined as follows. Draw AH to meet S again at X, then draw XK to meet S again at U. Here K is the symmedian point. V, W are similarly defined. Alternatively U, V, W are the points on the Hagge circle of K (which coincides with S) such that U, V, W are the reflections of D, E, F in BC, CA, AB respectively, where D, E, F are the intersections of AK, BK, CK with the circumcircle of ABC.

2. The orthocentroidal circle and the points U, V, W

The equation of the orthocentroidal circle S, see Bradley and Smith [1], is

$$(b^2 + c^2 - a^2)x^2 + (c^2 + a^2 - b^2)y^2 + (a^2 + b^2 - c^2)z^2 - a^2yz - b^2zx - c^2xy = 0. \qquad (2.1)$$

The equation of the line AH is

$$(c^2 + a^2 - b^2)y = (a^2 + b^2 - c^2)z. \qquad (2.2)$$

This meets S at the point X with co-ordinates $(a^2, a^2 + b^2 - c^2, c^2 + a^2 - b^2)$. The symmedian point K has co-ordinates (a^2, b^2, c^2) and so XK has equation

$$(b^2 + c^2 - a^2)(b^2 - c^2)x + a^2(a^2 - b^2)y + a^2(c^2 - a^2)z = 0. \qquad (2.3)$$

This meets the circle S again at U with co-ordinates $(a^2, b^2 + c^2 - a^2, b^2 + c^2 - a^2)$. Similarly V has co-ordinates $(c^2 + a^2 - b^2, b^2, c^2 + a^2 - b^2)$ and W has the co-ordinates $(a^2 + b^2 - c^2, a^2 + b^2 - c^2, c^2)$. (It will be seen these points are those called aH, bH, cH in Article 19.)

3. Circles BUC, CVA, AWB all pass through H

The equation of the circle BUC is

$$(b^2 + c^2 - a^2)x^2 - a^2yz + (c^2 - a^2)zx - (a^2 - b^2)xy = 0. \qquad (3.1)$$

It may now be checked that this circle passes through H and by cyclic change of x, y, z and a, b, c it follows that circles CVA and AWB also pass through H.

4. Circles BVC, CWA, AUB all pass through Ω

The equation of circle BVC is

$$b^2x^2 - a^2yz + (b^2 - c^2)xy = 0 \qquad (4.1)$$

and the equation of CWA is

$$c^2y^2 + (c^2 - a^2)yz - b^2zx = 0. \qquad (4.2)$$

These circles meet at the point $\Omega(1/b^2, 1/c^2, 1/a^2)$.

Similarly circles BWC. CUA, AVB meet at the point $\Omega'(1/c^2, 1/a^2, 1/b^2)$.

5. Circles BWU, CUV, AVW meet on the circumcircle of ABC

The equation of the circle BWU turns out to be

$$c^2(b^2 - c^2)(b^2 + c^2 - a^2)x^2 + a^2(a^2 - b^2)(a^2 + b^2 - c^2)z^2 + c^2a^2(b^2 - c^2)yz$$
$$- (a^6 - a^4(b^2 + c^2) - a^2(b^4 - 3b^2c^2 + c^4) + (b^2 + c^2)(b^2 - c^2)^2)zx + c^2a^2(a^2 - b^2)xy = 0. \qquad (5.1)$$

The equations of circles CUV and AVW may now be written down by cyclic change of x, y, z and a, b, c.

These circles all meet at L on the circumcircle of ABC with co-ordinates (x, y, z), where

$$x = a^2/((b^2 - c^2)(b^2 + c^2 - a^2)),$$
$$y = b^2/((c^2 - a^2)(c^2 + a^2 - b^2)), \qquad (5.2)$$
$$z = c^2/((a^2 - b^2)(a^2 + b^2 - c^2)).$$

Circles BUV, CVW, AWU meet at M and circles BVW, AWU, AUV meet at N, see the figure.

Reference

1. C.J.Bradley and G.C.Smith, *The locations of triangle centres*, Forum. Geom., 6 (2006) 57-70.

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