ARTICLE 29

The Miquel Circles for a Quadrilateral

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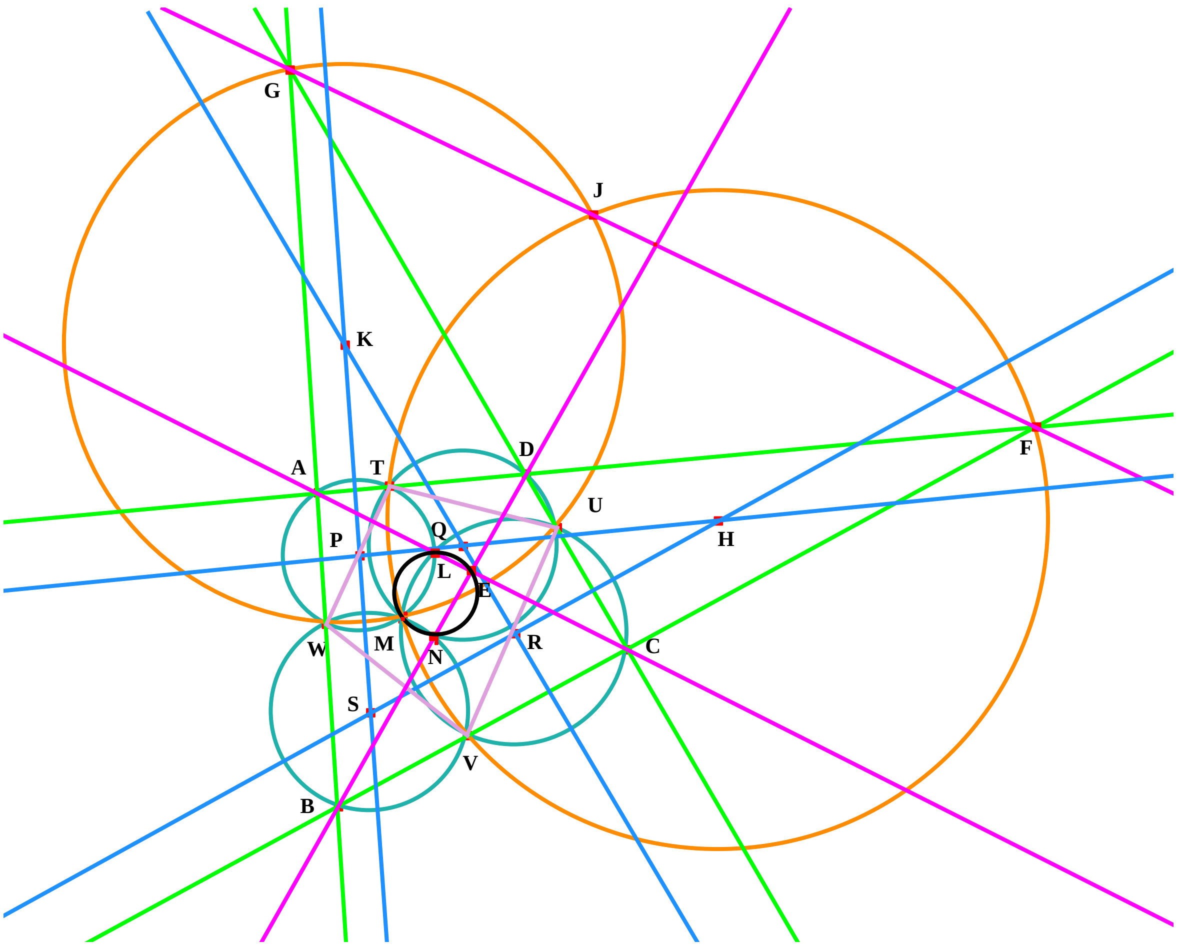

1. Introduction

Let ABCD be a quadrilateral in which no pair of opposite sides are parallel, and let AD and BC meet at F and BA and CD meet at G. In triangle BFG, points V, W, J are taken on BF, BG, FG respectively. Now let M be the Miquel point for this configuration, so that circles JFV, JGW, BVW meet at M. Suppose now circle JFV meets ADF at T and circle JGW meets GDC at U.

Now consider triangle FAB with T on FA, V on FB and W on AB. Since circles BVW and FJV are known to meet at M, it follows that M is the Miquel point for this configuration and it follows circle ATW also passes through M. Similarly by considering triangle GBC, it follows that circle CVU passes through M. The following results now hold:

(i) Circle DTU passes through M;
(ii) Circles ATW and CVU meet at M and a point L lying on AC;
(iii) Circles BWV and DTU meet at M and a point N lying on BD;
(iv) If E is the intersection of the diagonals AC and BD, then L, E, M, N are concyclic.

Whereas it is true that results (i), (ii), (iii) can be proved by synthetic arguments, there does not seem to be a pure argument of (iv). And unfortunately the analytic proof of (iv) requires an analysis of everything else first and this means a technically difficult proof. We use areal co-ordinates with B(1, 0, 0), F(0, 1, 0), G(0, 0, 1) so that BFG is triangle of reference.

2. Points J, C, D, A, M

We take C to be the point C(m, 1 – m, 0), A to be the point A(1 – n, 0, n) and J to be the point J(0, k, 1 – k). The equation of CG is

$$(1 - m)x = my, \tag{2.1}$$

and the equation of the line AF is

$$(1 - n)z = nx. \tag{2.2}$$

CG and AF meet at D(m(1 – n), (1 – m)(1 – n), mn). Rather than designate co-ordinates for points such as T, U, V, W, it is easier to let M have co-ordinates (r, s, t) and to find T, U, V, W in terms of r, s, t and other variables. Now since we are not dealing with normalised co-ordinates or the ratio of lengths we can choose r + s + t to be whatever we like, without altering the position of M. We therefore choose r + s + t = 1/k, so that J can be made to alter its position without M moving. This simplification still allows M and J to have independent positions.

3. Circles FJM, GJM and points T, U, V, W

In this section we give no working, but simply display the results as output by the algebraic computer package *DERIVE* that we used, the aim being to prevent errors in typing. The equation of the circle FJM is

$$-x^2(a^2t(s-1) + r(b^2t + c^2s)) - x(y(a^2t(s-1) + r(b^2t - c^2(r+t))) + z(a^2(r + st - t) - r(b^2(r+s) - c^2s))) + a^2rz(y(r+s+t-1) - z) = 0. \tag{3.1}$$

The circle GJM has equation

$$x^2(a^2s(r+s-1) - r(b^2t + c^2s)) + x(z(a^2s(r+s-1) + r(b^2(r+s) - c^2s)) - y(a^2(r^2 + r(t-1) - s^2 + s) + r(b^2t - c^2(r+t)))) - a^2ry(y(r+s+t-1) - z) = 0. \tag{3.2}$$

The co-ordinates of T are (x, y, z), where

$$\begin{aligned} x &= (n-1)(a^2(n(r^2 + r(s+t-1) + st - t) - st + t) + r(n-1)(b^2t - c^2(r+t))), \\ y &= r(1-n)(b^2(n(r+s+t) - t) - c^2s) - a^2(n(r - st + t) + st - t), \\ z &= -n(a^2(n(r^2 + r(s+t-1) + st - t) - st + t) + r(n-1)(b^2t - c^2(r+t))). \end{aligned} \tag{3.3}$$

The co-ordinates of V are (x, y, z), where

$$x = a^2t(1 - s) - r(b^2t - c^2(r + t)),$$
$$y = a^2t(s - 1) + r(b^2t + c^2s), \quad (3.4)$$
$$z = 0.$$

The co-ordinates of W are (x, y, z), where

$$x = a^2s(r + s - 1) + r(b^2(r + s) - c^2s),$$
$$y = 0, \quad (3.5)$$
$$z = r(b^2t + c^2s) - a^2s(r + s - 1).$$

The co-ordinates of U are (x, y, z), where

$$x = -m(a^2(m(r + s)(s - 1) + r) + mr(b^2(r + s) - c^2s)),$$
$$y = (m - 1)(a^2(m(r + s)(s - 1) + r) + mr(b^2(r + s) - c^2s)), \quad (3.6)$$
$$z = a^2(m(r^2 + r(2s + t - 1) + s(s - 1)) - r(r + s + t - 1)) - mr(b^2t + c^2(m(r + s + t) - r - t)).$$

4. **The circles through A, B, C, D**

In Section 3 we obtained points T, U, V, W lying on the lines AD, DC, CB, BA respectively. These points cannot be independent of one another, for if one places points on the sides at random then the four circles ATW, BWV, CVU, DUT do not pass through a common point. In other words the circles FJM, GJM form a recipe for obtaining four points T, U, V, W having the property that circles ATW, BWV, CVU, DUT share a common point M. In this way, for any quadrilateral without a pair of parallel sides, it is shown how to construct a Miquel point and the four associated circles.

In this section we find the equations of the four circles and check that they do share the common point M with circles FJM and GJM.

The equation of circle BWV is

$$x(y(a^2t(s - 1) + r(b^2t + c^2s)) + z(r(b^2t + c^2s) - a^2s(r + s - 1))) + y^2(a^2t(s - 1) + r(b^2t - c^2(r + t))) + yz(a^2(r^2 + rt + (1 - s)(s - t)) - r(b^2(r + s - t) + c^2(r - s + t))) - z^2(a^2s(r + s - 1) + r(b^2(r + s) - c^2s)) = 0. \quad (4.1)$$

The equation of circle DUT is

$$x^2(a^2(m(n(rs + (s - 1)(s - t)) + t(s - 1)) + t(n - 1)(s - 1)) - r(b^2t + c^2s)(m(2n - 1) - n + 1)) + x(z(a^2(m(n(r(2s - 1) + (s - 1)(2s - t)) + (1 - s)(r + s - t)) + (n - 1)(r + t(s - 1))) + r(b^2(m(n(2r + 2s - t) - r - s + t) - (n - 1)(r + s)) - c^2s(m(3n - 2) - n + 1))) - y(a^2(m(n(r^2 + r(t - 1) + (1 - s)(s - 2t)) - 2t(s - 1)) + t(1 - s)(n - 1)) + r(b^2t(m(3n - 2) - n + 1) + c^2((n - 1)(r + t) - m(n(2r - s + 2t) - r + s - t))))) - my^2(a^2(n(r^2 + r(s + t - 1) + t(s - 1)) - t(s - 1)) + r(n - 1)(b^2t - c^2(r + t))) + yz(a^2(m(n(rs + (s - 1)(s - t)) + (1 - s)(r + s - t)) - r(n - 1)(r + s + t - 1)) + mr(n - 1)(b^2(r + s - t) + c^2(r - s + t))) + z^2(n - 1)(a^2(m(r + s)(s - 1) + r) + mr(b^2(r + s) - c^2s)) = 0. \quad (4.2)$$

The equation of circle ATW is

$$nx^2 (a^2s(r + s - 1) - r(b^2 t + c^2 s)) + x(z(a^2s(r + s - 1)(2n - 1) + r(b^2(n(r + s - t) + t) + c^2s(1 - 2n))) - y(a^2(n(r^2 + r(t - 1) + (1 - s)(s - t)) - t(s - 1)) + r(b^2 t(2n - 1) - c^2 (n(r - s + t) + s)))) - y^2(a^2 (n(r^2 + r(s + t - 1) + t(s - 1)) - t(s - 1)) + r(n - 1)(b^2 t - c^2 (r + t))) - yz(a^2(n(r^2 + r(t - 1) + (1 - s)(s - t)) - r^2 - rt + (s - 1)(s - t)) + r(1 - n)(b^2 (r + s - t) + c^2 (r - s + t))) + z^2 (n - 1)(a^2 s(r + s - 1) + r(b^2 (r + s) - c^2 s)) = 0. \qquad (4.3)$$

The equation of circle CUV is

$$x^2 (m - 1)(a^2 t(s - 1) + r(b^2 t + c^2s)) + x(y(a^2 t(s - 1)(2m - 1) + r(b^2t(2m - 1) - c^2(m(r - s + t) - r - t))) - z(a^2 (m(r + s - t)(s - 1) + r + t(s - 1)) + r(b^2 (m(r + s - t) - r - s) + c^2 s(1 - 2m)))) + my^2(a^2t(s - 1) + r(b^2t - c^2(r + t))) - yz(a^2(m(r + s - t)(s - 1) - r(r + s + t - 1)) + mr(b^2(r + s - t) + c^2(r - s + t))) - z^2(a^2(m(r + s)(s - 1) + r) + mr(b^2(r + s) - c^2s)) = 0. \qquad (4.4)$$

It may now be checked that all four circles pass through M(r, s, t).

5. The points N, L, E and the circle LENM

The point N lies on BD and is the intersection of circles DUT and BVW. Its co-ordinates are (x, y, z), where

$$x = a^2(m^2 (n^2 (r^2 + r(t - s) + 2(1 - s)(s - t)) - n(r^2 + rt + (1 - s)(s - 3t)) + t(s - 1)) + m(1 - n)(n(r^2 + rt + (1- s)(s - 3t)) - 2t(s - 1)) + t(n - 1)^2 (s - 1)) - r(b^2(m(n(r + s - t) + t) + t(n - 1)) + c^2(m(n(r - s + t) - r - t) - (n - 1)(r + t)))(m(2n - 1) - n + 1),$$

$$y = (m - 1)(n - 1)(a^2(m(n(rs + (s - 1)(s - t)) + t(s - 1)) + t(n - 1)(s - 1)) - r(b^2t + c^2s)(m(2n - 1) - n + 1)), \qquad (5.1)$$

$$z = mn(a^2(m(n(rs + (s - 1)(s - t)) + t(s - 1)) + t(n - 1)(s - 1)) - r(b^2t + c^2s)(m(2n - 1) - n + 1)).$$

The point L lies on AC and is the intersection of circles ATW and CUV. Its co-ordinates are (x, y, z), where

$$x = a^2(m^2(s - 1)(n(r + s + t) - t) + m(n^2(r^2 + r(2s + t - 1) + s^2 + s(t - 1) - t) - n(r^2 + r(2s + t - 2) + s^2 + s(3t - 1) - 3t) + 2t(s - 1)) - n^2(r^2 + r(s + t - 1) + t(s - 1)) + n(r^2 + r(s + t - 1) + 2t(s - 1)) - t(s - 1)) + r(m + n - 1)(b^2(m(n(r + s + t) - t) - t(n - 1)) + c^2((n - 1)(r + t) - m(n(r + s + t) - r - t))),$$
$$y = (1 - m)(a^2(m(s - 1)(n(r + s + t) - t) + n(r - t(s - 1)) + t(s - 1)) + r(m + n - 1)(b^2(n(r + s + t) - t) - c^2s)), \qquad (5.2)$$
$$z = n(r(m + n - 1)(b^2t + c^2(m(r + s + t) - r - t)) - a^2(m(n(r^2 + r(2s + t - 1) + (s - 1)(s + t)) - t(s - 1)) - n(r^2 + r(s + t - 1) + t(s - 1)) + t(s - 1))).$$

The point E is the intersection of the diagonals AC and BD and has co-ordinates (x, y, z), where

$$x = 2n(1 - n),\quad y = (1 - m)(1 - n),\quad z = mn. \qquad (5.3)$$

It may now be verified that points L, E, N, M all lie on the circle with equation

$$x^2(a^2(m(n(rs + s^2 - s(t + 1) + t) + t(s - 1)) + t(n - 1)(s - 1)) - r(b^2t + c^2s)(m(2n - 1) - n + 1)) + x(z(a^2(m(n(r(3s - 1) + 3s^2 - s(t + 3) + t) + r(1 - 2s) + (1 - s)(2s - t)) + (n - 1)(r + t(s - 1))) + r(b^2(m(2n(r + s - t) - r - s + 2t) - (n - 1)(r + s)) - c^2s(m(4n - 3) - n + 1))) - y(a^2(m(n(r^2 + r(t - 1) - s^2 + s(3t + 1) - 3t) - 3t(s - 1)) + t(1 - s)(n - 1)) + r(b^2t(m(4n - 3) - n + 1) + c^2((n - 1)(r + t) - m(2n(r - s + t) - r + 2s - t)))))) - my^2(a^2(n(r^2 + r(s + t - 1) + 2t(s - 1)) - 2t(s - 1)) + 2r(n - 1)(b^2 t - c^2(r + t))) - yz(a^2(m(n(r^2 + r(t - s) + 2(1 - s)(s - t)) - r^2 + r(s - t - 1) + 2(s - 1)(s - t)) + r(n - 1)(r + s + t - 1)) + 2mr(1 - n)(b^2(r + s - t) + c^2(r - s + t))) + z^2(n - 1)(a^2(m(r(2s - 1) + 2s(s - 1)) + r) + 2mr(b^2 (r + s) - c^2 s)) = 0. \qquad (5.4)$$

6. Other quadrilaterals

The same construction of the Miquel point is valid with minor simplifications for quadrilaterals that have one or two pairs of opposite sides parallel. For a trapezium with AD parallel to BC, F recedes to infinity and the circle FJM becomes a line. For a parallelogram both F and G recede to infinity and both circles FJM and GJM become lines. The result is that TMV and WMU become straight lines.

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