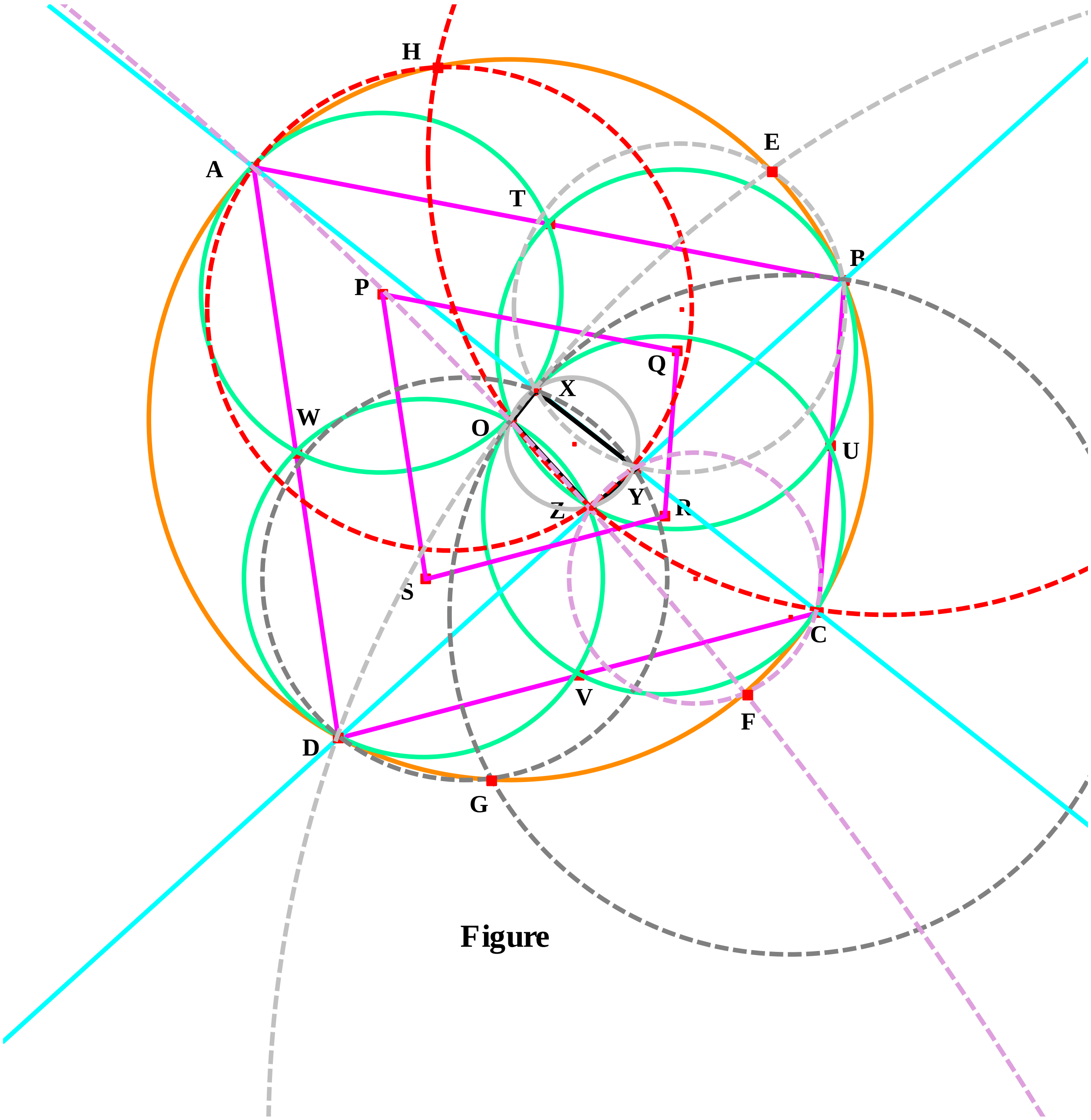
Article 28

Some Circles in a Cyclic Quadrilateral

Christopher J Bradley



Figure

1. Introduction

In the figure ABCD is a cyclic quadrilateral, Points T, U, V, W are the midpoints of AB, BC. CD, DA respectively and O is the centre, X is the midpoint of AC and Z is the midpoint of BD. The diagonals AC and BD meet at Y. Points P, Q, R, S are the midpoints of OA, OB, OC, OD respectively, so that PQRS is a cyclic quadrilateral similar to ABCD and half the size. The following results now hold, several of which are immediate and others more interesting.

(i) Circles ATW, BUT, CVU, DWV all pass through O and touch the circle ABCD at its vertices A, B, C, D respectively;
(ii) Circles ATW, CVU pass through X and circles BUT, DWV pass through Z;
(iii) Points O, X, Y, Z are concyclic;
(iv) Circles AYZ and COZ meet at Z and a point H lying on the circumcircle;
(v) Circles CYZ and AOZ meet at Z and a point F lying on the circumcircle;
(vi) Circles BXY and DOX meet at X and a point E lying on the circumcircle;
(vii) Circles DXY and BOX meet at X and a point G lying on the circumcircle.
(viii) Angle OXY = angle OZY = 90^o.

In the following sections we prove these results using Cartesian co-ordinates.

2. Result (i)

Take ABCD as the unit circle with O the origin. Let A have co-ordinates $(2a/(1 + a^2), (1 – a^2)/(1 + a^2)$ and let B. C, D have similar co-ordinates with parameters b, c, d respectively. The co-ordinates of T, the midpoint of AB, are therefore $\{(1 + ab)/((1 + a^2)(1 + b^2))\}(a + b, 1 – ab)$. Similar expressions hold for the co-ordinates of U, V, W by making appropriate changes of parameters.

The equation of the circle ATW may now be calculated and is

$$(1 + a^2)(x^2 + y^2) – 2ax – (1 – a^2)y = 0. \qquad (2.1)$$

Note that this circle passes through the point O and has centre P, the midpoint of AO. Similarly the circles BUT, CVU, DWV all pass through O and have centres at Q, R, S respectively. An alternative argument for this result is that the four circles are the Miquel circles for the four vertices A, B, C, D and must have a common point. And furthermore the quadrangle of centres must be similar to the quadrangle ABCD, and since it derives from the midpoints of chords the similarity is simply a reduction by a factor of 2 through the Miquel point O.

It is also easy to check that circle ATW touches circle ABCD at A.

3. Result (ii)

The circle CVU has an equation similar to Equation (2.1) with c replacing a. It follows trivially that the circles ATW, CVU meet at the midpoint X of AC as well as at O. Similarly circles BUT and DWV meet at the midpoint Z of BC as well as at O. The co-ordinates of X and Z are similar to those of T but with appropriate changes of parameters.

4. Result (iii)

The point Y is the intersection of the diagonals AC and BD. The line AC has equation

$$(a + c)x + (1 – ac)y = 1 + ac. \quad (4.1)$$

The line BD is similar with b, d replacing a, c. These two lines meet at Y with co-ordinates (x, y), where

$$x = 2(ac – bd)/(abc + acd – abd – bcd + a – b + c – d), \quad (4.2)$$
$$y = (abc + acd – abd – bcd – a + b – c + d)/(abc + acd – abd – bcd + a – b + c – d). \quad (4.3)$$

Having obtained the co-ordinates of X, Y, Z we may now obtain the equation of circle XYZ and this is found to be

$$(abc + acd – abd – bcd + a – b + c – d)(x^2 + y^2) + 2(bd – ac)x + (abc + acd – abd – bcd – a + b – c + d)y = 0. \quad (4.4)$$

Clearly this circle passes through O.

5. Result (iv)

The circle AZY has equation

$$(abc + acd – abd – bcd + a – b + c – d)(x^2 + y^2) + (ab + ad – 2ac + 2bd – bc – cd)x + (abc – 2abd + acd + b – 2c + d)y – (a – c)(1 + bd) = 0. \quad (5.1)$$

The circle COZ has equation

$$(bc^2 – 2bcd + c^2d – b + 2c – d)(x^2 + y^2) + 2x(bd – c^2) + (bc^2 – 2bcd + c^2d + b – 2c + d) = 0. \quad (5.2)$$

These circles meet at Z and the point H with co-ordinates (x, y), where

$$x = – 2\{(b – 2c + d)(bc + cd – 2bd)\}/\{b^2(c^2 – 4cd + 4d^2 + 1) + 2b(c^2d – 2c(1 + d^2) + d) + c^2(4 + d^2) – 4cd + d^2\}, \quad (5.3)$$

$$y = – \{(b(c – 2d + 1) + c(d – 2) + d)(b(c – 2d – 1) + c(d + 2) – d)\}/)\}/\{b^2(c^2 – 4cd + 4d^2 + 1) + 2b(c^2d – 2c(1 + d^2) + d) + c^2(4 + d^2) – 4cd + d^2\}. \quad (5.4)$$

It may now be verified that (x, y) given by Equations (5.3) and (5.4) satisfy $x^2= y^2 = 1$ so that circles AZY and COZ intersect at Z and a point H on the circumcircle of ABCD. Results (v), (vi), (vii) now follow by symmetry.

6. Result (viii)

It is straightforward to show the co-ordinates of the centre K of circle OXYZ are half those of point Y, which proves that O, K, Y are collinear and hence angles OZY and OXY are right angles.

7. The cyclic pentagon

In Figure 2 we show a cyclic pentagon, in which the same construction is carried out for each of the five component cyclic quadrilaterals. There are five diagonals whose midpoints are labelled X_1 › X_5 and whose intersections (other than A, B, C, D, E) are labelled Y_1› Y_5. There are thus 5 Miquel circles touching the circumcircle at its vertices and 5 more circles such as $OX_1Y_2X_2$. There would also be 40 circles intersecting in pairs on the circumcircle. For clarity we show only one of these pairs in Figure 2, AY_2X_2^COX_2 = H_2.

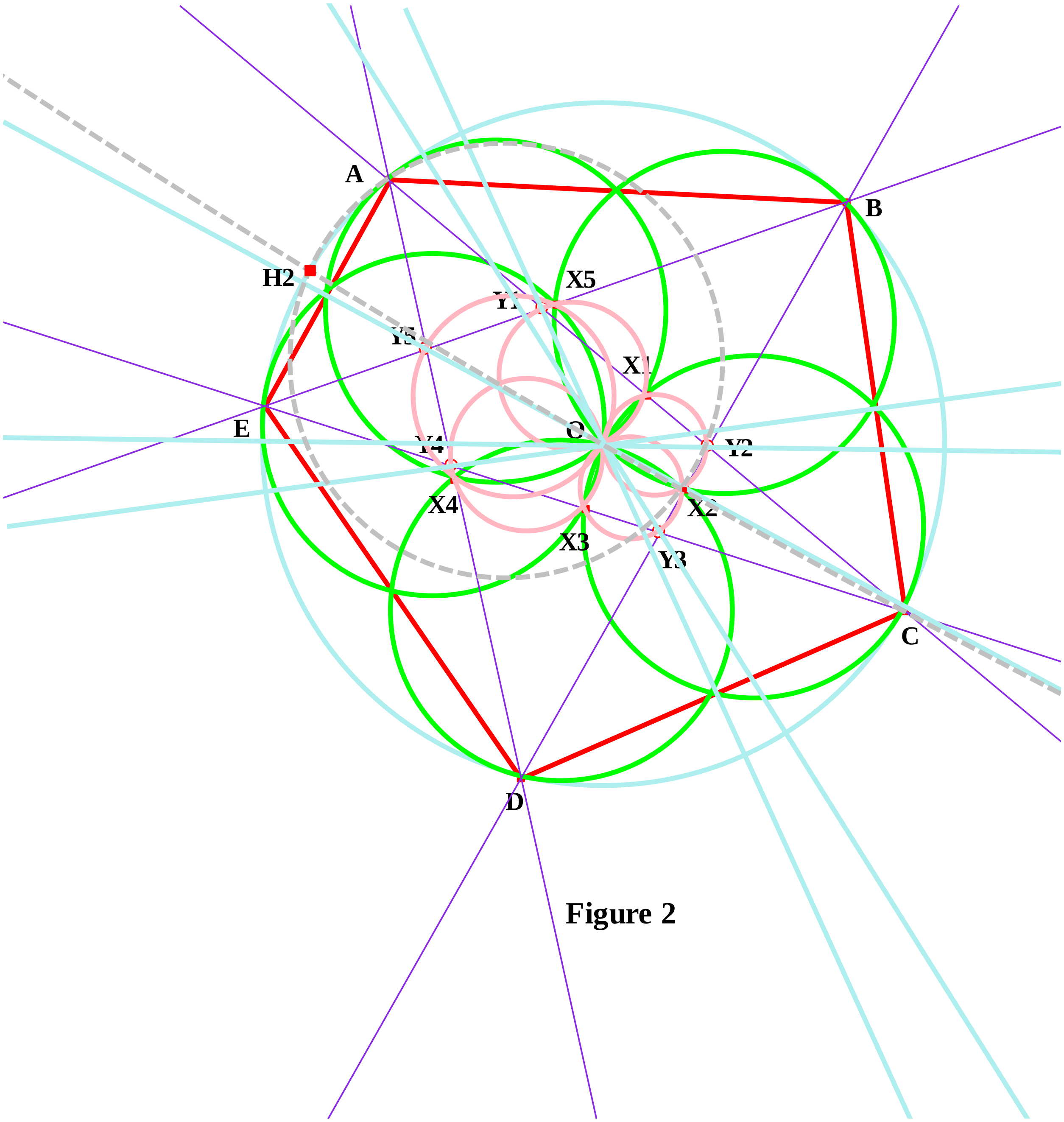


Figure 2

Flat 4,
Terrill Court,
12-14 Apsley Road,
BRISTOL BS8 2SP.