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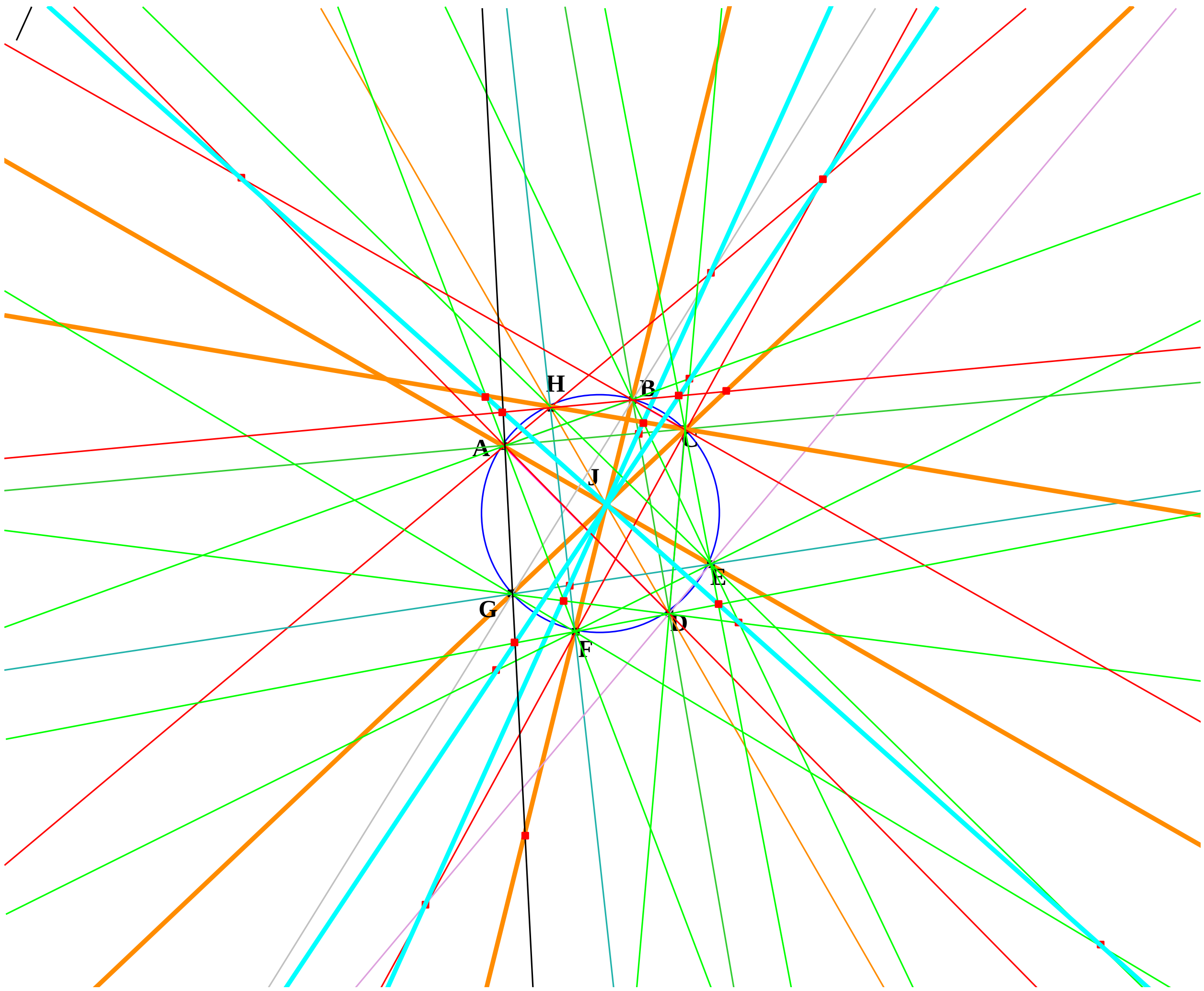
Quadrangles in Perspective Part II   The three lines

Christopher Bradley

Abstract: Two quadrangles are drawn in a conic and are in perspective. Three lines emerge with six points of intersection on each line. They are like Pascal lines but appropriate to Quadrilaterals in perspective and not to pairs of triangles. It is difficult to be sure but they may be a new discovery.



The three blue lines

1. Introduction

ABCD and EFGH are two quadrangles inscribed in a conic (not just a circle as in the diagram). Various points of intersection to be detailed later are found to exist, six points of intersection on each of three special lines. These lines are similar to Pascal lines but are not related to them and are only three in number. We found them something of a surprise.

We take the conic to have equation

$$y^2 = zx, \qquad (1.1)$$

and the vertex of perspective to be J(0, 1, 0). These choices allow us to choose the co-ordinates of A to be A(a^2, a, 1) etc and those of E to be (a^2, – a, 1), etc. Parameters b, c, d are used for other pairs of points in the perspective.

The equation of BC is found to be

$$x – (b + c)y + abz = 0, \qquad (1.2)$$

and similarly for AD. These intersect at the point BC^AD with co-ordinates
BC^AD: (ad(b + c) – bc(a + d), ad – bc, a – b – c + d).

2. Other points of intersection

The co-ordinates of other points of intersection may be deduced by change of co-ordinates to a, b, c, d, – a, – b, – c, – d as appropriate.

BE^GD: (acd + bcd – abc – bcd, ab – cd, a – b – c + d), (2.1)
CE^FD: (acd + abd – abc – bcd, ac – bd, a – b – c + d), (2.2)
AG^HB: (acd + abd – abc – bcd, bd – ac, a – b – c + d), (2.3)
AF^HC: (acd + abd – abc – bcd, cd – ab, a – b – c + d), (2.4)
BC^AD: (adb + acd – abc –bcd, cad - bc, a – b – c + d), (2.5)
FG^HE: (abc + abd – acd – bcd, cd – ab, a + b – c – d). (2.6)

These six points lie on the line with equation

$$(a – b – c + d)x + (bcd + abc – abd – acd)z = 0. \qquad (2.7)$$

If you write (ADFG/BCEH) you can read off the points of intersection following a fairly obvious pattern. And there are two other lines in which points A, B, C, D, and E, F, G, H are permuted. If the above line is designated by BC^AD then the others are designated by CD^BA and CA^BD.

The other lines involve the following sets of 6 points:
(AB^CD, AH^FC, AG^FD, EF^GH, AG^FD, GB^ED),
(AH^BG, AF^GD, GE^HF, BD^AC, BE^HC, DE^FC).

Flat 4, Terrill Court, 12-14, Apsley Road, BRISTOL BS8 2SP.