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Perspective Quadrilaterals

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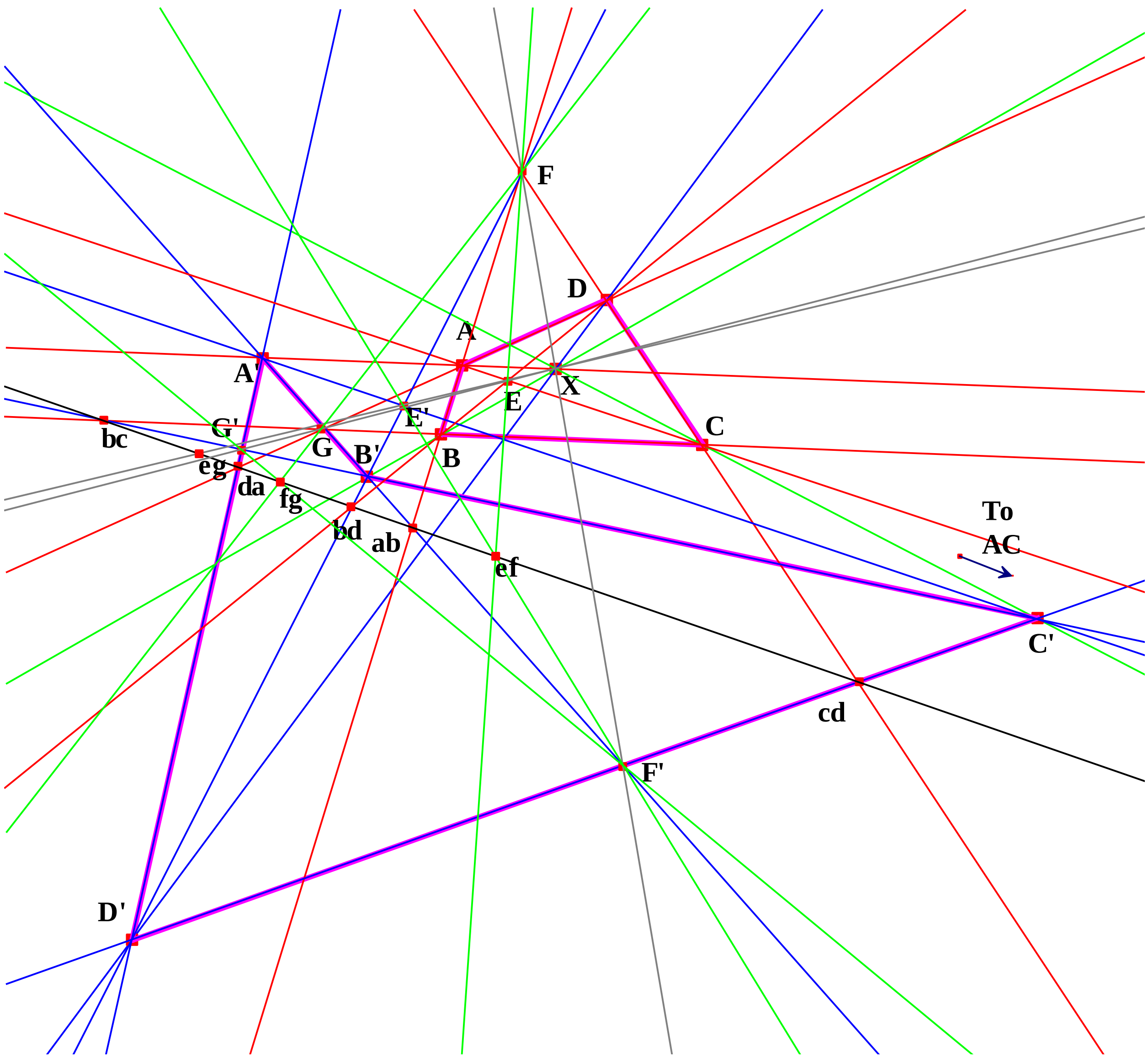
Abstract: Two quadrilaterals ABCD and A'B'C'D' are such that AA', BB', CC', DD' are concurrent at a point X. It is found that a single condition is such that the four Desargues' axes of perspective coincide. It is also the case that when this happens the diagonal point triangles EFG and E'F'G' are also in perspective with vertex of perspective X and the same Desargues' axis of perspective. See also Article 238.



Figure

1. The vertices of the two quadrilaterals

We suppose that ABC is the triangle of reference and we use projective co-ordinates throughout. Take D to have co-ordinates D(a, b, c). Let X be the vertex of perspective with co-ordinates X(1, 1, 1). There then exist constants l, m, n, k so that the co-ordinates of A', B', C', D' are as follows:

A'(1 + l, l, l), (1.1)

B'(m, 1 + m, m), (1.2)

C'(n, n, 1 + n), (1.3)

D'(a + k, b + k, c + k). (1.4)

2. Corresponding sides

The equation of AB is z = 0. The equation of A'B' is

$$lx + my = (l + m + 1)z. \quad (2.1)$$

The equation of BC is x = 0. The equation of B'C' is

$$my + nz = (m + n + 1)x. \quad (2.2)$$

Line AB meets line A'B' at ab with co-ordinates (– m, l, 0) and the line BC meets the line B'C' at bc with co-ordinates (0, – n. m). ab and bc lie on the line

$$lx + my + nz = 0 \quad (2.3)$$

and this we take to be the equation of the Desargues' axis, though one must expect one or more conditions for the meets of other pairs of corresponding sides to lie on this line.

The equation of CD is ay = bx and the equation of C'D' is

$$(b(n + 1) – cn + k)x + n(a – b)z = (a(n + 1) – cn + k)y. \quad (2.4)$$

Line CD meets line C'D' at cd with co-ordinates (an, bn, cn – k). The condition that cd lies on ab bc is

$$k = al + bm + cn. \quad (2.5)$$

It turns out that condition (2.5) is sufficient for all other pairs of corresponding sides to meet on ab bc. That is the four Desargues' axes coincide when the parameter k satisfies Equation (2.5). Geometrically this means there is one and only one position for D' on DX for which the four Desargues' axes coincide. For example ad has co-ordinates (al – k, bl, cl) and (2.5) is the condition for this point to lie on the line ab bc.

3. The diagonal point triangle

The equation of AC is y = 0 and the equation of BD is cx = az. The diagonal point E = AC^BD therefore has co-ordinates (a, 0, c).

The equation of A'C' is

$$lx + nz = (l + n + 1)y, \qquad (3.1)$$

and the equation of B'D' is

$$(bm - c(m + 1) - k)x + m(c - a)y + (a(m + 1) - bm + k)z. \qquad (3.2)$$

These two lines meet at the point E' with co-ordinates (x, y, z), where

$$x = a(l(m + 1) + m + n + 1) - bm(n + l + 1) + cmn + k(n + l + 1), \qquad (3.3)$$
$$y = al(m + 1) - bm(n + l) + cn(m + 1) + k(n + l), \qquad (3.4)$$
$$z = alm - bm(n + l + 1) + c(l + (m + 1)(n + 1)) + k(n + l + 1). \qquad (3.5)$$

The equation of EP is

$$az + y(c - a) = cx. \qquad (3.6)$$

It is now found that the condition that E' lies on EP is the condition (2.5).

Now AB has equation z = 0 and CD has equation bx = ay. The diagonal point F = AB^CD therefore has co-ordinates (a, b, 0). The line A'B' has equation

$$lx + my = (l + m + 1)z, \qquad (3.7)$$

and the equation of C'D' is

$$(b(n + 1) - cn + k) + n(a - b)z = (a(n + 1) - cn + k)y. \qquad (3.8)$$

The diagonal point F' = A'B'^C'D' has co-ordinates

$$x = a(l(n + 1) + m + n + 1) + bmn - (l + m + 1)(cn - k), \qquad (3.9)$$
$$y = anl + b(l + (m + 1)(n + 1)) - (l + m + 1)(cn - k), \qquad (3.10)$$
$$z = al(n + 1) + bm(n + 1) - (l + m)(cn - k). \qquad (3.11)$$

The equation of FP is

$$bx + (a - b)z = ay. \qquad (3.12)$$

and the condition that F' lies on FP is once again Equation (2.5).

It is now a matter of some heavy technical work to establish that ef = EF^E'F' lies on the axis ab bc.

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