ARTICLE 24

The Thirteen point Circle

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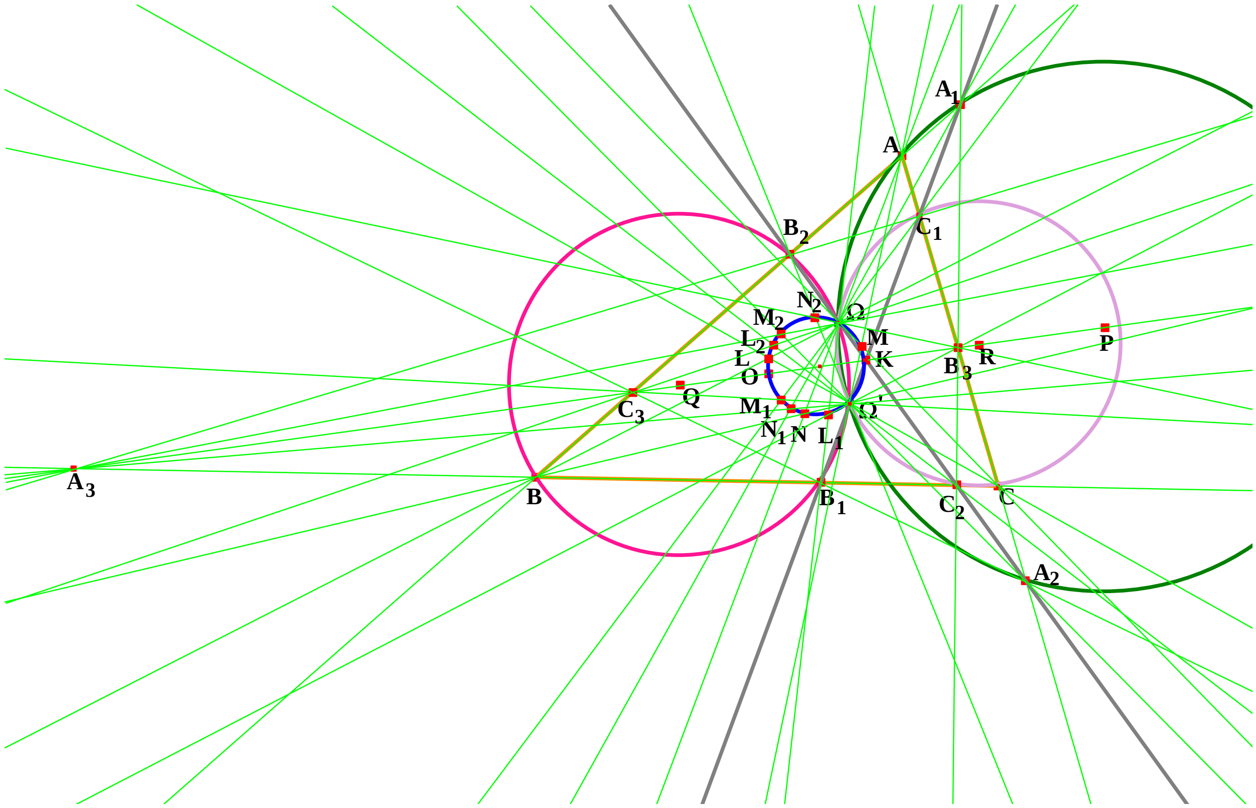

1. Introduction

In this article we give an account of the properties of the coaxal system of circles passing through the two Brocard points Ω and Ω' and having the Brocard axis as line of centres. This coaxal system contains four circles of particular importance, the seven point circle S and the three circles AΩΩ', BΩΩ' and CΩΩ', where A, B and C are the vertices of the triangle. The circumcircle is not a member of this coaxal system, nor is the coaxal system orthogonal to the coaxal system containing the Apollonius circles.

We now introduce our notation. The centres of the four circles just mentioned are (i) the midpoint of OK, where O is the circumcentre and K the symmedian point, (ii) P, (iii) Q, and (iv) R. The circle AΩΩ' meets AB at A_1 and CA at A_2. The circle BΩΩ' meets BC at B_1 and AB at B_2. The circle CΩΩ' meets CA at C_1 and BC at C_2. A_1Ω meets S at N_1, B_1Ω meets S at L_1, C_1Ω meets S at M_1, $A_2\Omega'$ meets S at M_2, $B_2\Omega'$ meets S at N_2 and $C_2\Omega'$ meets S at L_2. These additional points on S convert it from being the seven point circle to the thirteen point circle of the title.

We establish the following results:

(i) P, Q, R lie on the Brocard axis;

(ii) BCB_1C_2 meets B_2C_1 at a point A_3 on the Brocard axis, CAC_1A_2 meets C_2A_1 at a point B_3 on the Brocard axis and ABA_1B_2 meets A_2B_1 at a point C_3 on the Brocard axis;

(iii) $A_1B_1C_1$ is a straight line passing through Ω' and $A_2B_2C_2$ is a straight line passing through Ω and these two lines intersect at K, the symmedian point;

(iv) $M_2\Omega$ and $N_1\Omega'$ pass through A_3, $N_2\Omega$ and $L_1\Omega'$ pass through B_3, and $L_2\Omega$ and $M_1\Omega'$ pass through C_3.

These results are illustrated in the Figure generated by *CABRI* software and the computer algebra package *DERIVE* was used to check the algebra for which areal co-ordinates are used throughout.

2. The seven point circle and the Brocard axis

The Brocard axis passes through O and K and putting (x, y, z) and their co-ordinates as rows of a determinant we find its equation to be

$$\frac{x(b^2-c^2)}{a^2}+\frac{y(c^2-a^2)}{b^2}+\frac{z(a^2-b^2)}{c^2}=0 \qquad (2.1)$$

In areal co-ordinates the equation of any circle may be put in the form

$$a^2yz+b^2zx+c^2xy-(x+y+z)(ux+vy+wz)=0, \qquad (2.2)$$

where u, v, w are constants to be determined, see Bradley [1, 2]. Putting in the co-ordinates of $K(a^2, b^2. c^2)$, $\Omega(1/b^2, 1/c^2, 1/a^2)$, $\Omega'(1/c^2, 1/a^2, 1/b^2)$ we obtain three equations to determine u, v, w and after some simplifications we obtain the equation of the seven point circle in the form

$$b^2c^2x^2+c^2a^2y^2+a^2b^2z^2-a^4yz-b^4zx-c^4xy=0. \qquad (2.3)$$

3. The three circles and their intersections with the sides of the triangle

We obtain the equations of the circles $A\Omega\Omega'$, $B\Omega\Omega'$ and $C\Omega\Omega'$ using the same method and their equations turn out to be

$$\frac{y^2(b^2-a^2)}{b^2}+\frac{z^2(c^2-a^2)}{c^2}+\frac{yz(a^2-b^2)(a^2-c^2)}{b^2c^2}+\frac{zx(a^2-b^2)}{a^2}+\frac{xy(a^2-c^2)}{a^2}=0, \qquad (3.1)$$

$$\frac{z^2(c^2-b^2)}{c^2}+\frac{x^2(a^2-b^2)}{a^2}+\frac{zx(b^2-c^2)(b^2-a^2)}{c^2a^2}+\frac{xy(b^2-c^2)}{b^2}+\frac{yz(b^2-a^2)}{b^2}=0, \qquad (3.2)$$

$$\frac{x^2(a^2-c^2)}{a^2}+\frac{y^2(b^2-c^2)}{b^2}+\frac{xy(c^2-a^2)(c^2-b^2)}{a^2b^2}+\frac{yz(c^2-a^2)}{c^2}+\frac{zx(c^2-b^2)}{c^2}=0, \qquad (3.3)$$

respectively. The intersections of these circles with the sides of ABC are now easily found and are:
$A_1(a^2(a^2 – b^2), b^2(a^2 – c^2), 0)$, $A_2(a^2(a^2 – c^2), 0, c^2(a^2 – b^2))$, $B_1(0, b^2(b^2 – c^2), c^2(b^2 – a^2))$, $B_2(a^2(b^2 – c^2), b^2(b^2 – a^2), 0)$, $C_1(a^2(c^2 – b^2), 0, c^2(c^2 – a^2))$, $C_2(0, b^2(c^2 – a^2), c^2(c^2 – b^2))$

Since the perpendicular bisector of ΩΩ' is the Brocard axis it is clear that the centres P, Q, R of the circles AΩΩ', BΩΩ' and CΩΩ' all lie on the Brocard axis.

4. The lines $A_1B_1C_1$ and $A_2B_2C_2$

Using the usual determinantal method the equation of the line $A_1B_1C_1$ is

$$b^2c^2(a^2 - c^2)x + c^2a^2(b^2 - a^2)y + a^2b^2(c^2 - b^2)z = 0 \qquad (4.1)$$

and it may be checked that $K(a^2, b^2, c^2)$ and $\Omega'(1/c^2, 1/a^2, 1/b^2)$ both lie on this line. Similarly the equation of the line $A_2B_2C_2$ is

$$b^2c^2(a^2 - b^2)x + c^2a^2(b^2 - c^2)y + a^2b^2(c^2 - a^2)z = 0 \qquad (4.2)$$

And it may be checked that $K(a^2, b^2, c^2)$ and $\Omega(1/b^2, 1/c^2, 1/a^2)$ both lie on this line. Note that the two lines intersect at K on the Brocard axis.

5. The points A_3, B_3, C_3

The equation of C_1B_2 is

$$\frac{x}{a^2(b^2 - c^2)} + \frac{y}{b^2(a^2 - b^2)} + \frac{z}{c^2(c^2 - a^2)} = 0\ , \qquad (5.1)$$

and the equation of B_1C_2 is x = 0. These meet at the point $A_3(0, b^2(a^2 – b^2), c^2(a^2 – c^2))$, a point that clearly lies on the Brocard axis with Equation (2.1). Similarly B_3 and C_3 have co-ordinates $B_3(a^2(b^2 – a^2), 0, c^2(b^2 – c^2))$ and $C_3(a^2(c^2 – a^2), b^2(c^2 – b^2), 0)$ both of which also lie on the Brocard axis.

6. The six points L_1, L_2, M_1, M_2, N_1, N_2

For the record L = BΩ^CΩ', M = CΩ^ AΩ', N = AΩ^BΩ' have co-ordinates $L(a^2, c^2, b^2)$, $M(c^2, b^2, a^2)$, $N(b^2, a^2, c^2)$ and together with O, K, Ω, Ω' form the seven points of the seven point (Brocard) circle. As stated in Section 1 the six additional points to provide the thirteen in the Figure are such that $A_1\Omega$ meets S at N_1, $B_1\Omega$ meets S at L_1, $C_1\Omega$ meets S at M_1, $A_2\Omega'$ meets S at M_2, $B_2\Omega'$ meets S at N_2 and $C_2\Omega'$ meets S at L_2. After some algebra we find their co-ordinates to be $L_1(a^2b^2, b^2(c^2 + a^2 – b^2), b^4 + (a^2 – b^2)(c^2 + a^2))$, $M_1(c^4 + (b^2 – c^2)(a^2 + b^2), b^2c^2, c^2(a^2+ b^2 – c^2))$, $N_1(a^2(b^2 + c^2 – a^2), a^4 + (c^2 – a^2)(b^2 + c^2), c^2a^2)$, $L_2(c^2a^2, a^4 + (a^2 – c^2)(b^2 – c^2), c^2(a^2 + b^2 – c^2))$, $M_2(a^2(b^2 + c^2 – a^2), a^2b^2, b^4 + (b^2 – a^2)(c^2 – a^2))$, $N_2(c^4 + (c^2 – b^2)(a^2 – b^2), b^2(c^2 + a^2 – b^2), b^2c^2)$.

7. **More lines through A_3, B_3, C_3**

We give the analysis to show that $M_2\Omega$^$N_1\Omega'$ = A_3, the pairs of lines that intersect in B_3 and C_3 then follow by cyclic change of letters A, B, C and L, M, N.

The equation of $M_2\Omega$ is

$$b^2(a^4 - a^2(b^2 + c^2) + b^4)x + c^2a^2(c^2 - a^2)y + a^2b^2(a^2 - b^2)z = 0. \qquad (7.1)$$

The equation of $N_1\Omega'$ is

$$c^2(a^4 - a^2(b^2 + c^2) + c^4)x + c^2a^2(a^2 - c^2)y + a^2b^2(b^2 - a^2)z = 0. \qquad (7.2)$$

These two lines intersect at $A_3(0, b^2(a^2 - b^2), c^2(a^2 - c^2))$.

References

1. C. J. Bradley, *Challenges in Geometry*, Oxford, 2005.
2. C. J. Bradley, *The Algebra of Geometry*, Highperception, Bath, 2007.

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